P. CLÉMENT AND G. SWEERS

On subsolutions to a semilinear elliptic problem

ABSTRACT: The relation between the existence of a subsolution for the problem $-\Delta u = f(u)$ with 0 — Dirichlet boundary value on a bounded domain and on a ball of R^N is considered. As a consequence a necessary condition for the existence of solutions when f changes sign is given.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we consider the following problem:

$$\begin{aligned} -\Delta u &= f(u) \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of R^N, $N > 1$, and the function $f: R \to R$ is assumed to be only continuous.

We call a function $u \in C(\bar{\Omega})$ a "*subfunction on* Ω" if the following differential inequality

$$\int_\Omega [u(-\Delta\phi) - f(u)\phi]dx \leq 0 \tag{1.2}$$

holds for every $\phi \in D^+(\Omega)$, where $D^+(\Omega)$ consists of all nonnegative functions in $C_0^\infty(\Omega)$.

We shall make use of the following result (see [8, Lemma A.4, p. 105]): let u and v be subfunctions on Ω, then max(u,v) is also a subfunction on Ω. If the function u also satisfies

$$u \leq 0 \text{ on } \partial\Omega, \tag{1.3}$$

then we shall call u a "*subsolution on* Ω". If the inequality in (1.2) (resp. (1.2) and (1.3)) is reversed, then we call u a superfunction on Ω (resp. a supersolution on Ω). A *solution on* Ω is a function u which is both a sub- and a supersolution on Ω. We shall use the following notation:

for $x_0 \in \mathbb{R}^N$ and $R > 0$, $B(x_0,R) := \{x \in \mathbb{R}^N;\ |x-x_0| < R\}$ and $B_R = B(0,R)$.

THEOREM 1: Let u be a subsolution on Ω satisfying $\max_\Omega u > 0$. Then there exist $R > 0$ and a subsolution v on B_R with $\max_{B_R} v = \max_\Omega u$, satisfying:

(i) v is positive on B_R and $v = 0$ on ∂B_R,

(ii) v is radially symmetric.

REMARK: The function v can even be chosen such that $|x| \to v(x)$ is non-increasing on $[0,R]$. As a consequence, we have

THEOREM 2: If u is a subsolution on Ω with $\max_\Omega u > 0$, then

$$\int_s^{\max u} f(t)\,dt > 0 \text{ holds for all } s \in [0,\max u). \tag{1.4}$$

A first result in this direction was obtained by De Figueiredo in [6]. With additional regularity on f and for *positive* solutions, condition (1.4) has been proved to be necessary in Dancer and Schmitt [5]. See also for related results [2], [3], [9]. In the proof of Theorems 1 and 2, we avoid the use of a theorem of Gidas, Ni and Nirenberg [7], which requires the positivity of u and more regularity on f.

Concerning a partial converse of Theorem 1, we mention the following result [10]: Let v be a subsolution on B_1 satisfying $\max_{B_1} v > 0$. Suppose either $f(0) \geq 0$ or Ω satisfies a uniform interior sphere condition [1]. Then there exist $\lambda > 0$ and $u \in C(\bar\Omega)$ satisfying

$$\int_\Omega u(-\Delta\phi)dx \leq \lambda \int_\Omega f(u)\phi\,dx, \text{ for every } \phi \in D^+(\Omega),\ \max u = \max v,$$

$$u \text{ is positive on } \Omega, \text{ and} \tag{1.5}$$

$$u = 0 \text{ on } \partial\Omega.$$

If $f(0) < 0$ and Ω does not satisfy a uniform sphere condition, it may happen that the conclusion fails.

EXAMPLE: Let $f(u) = -\cos u$. Then if the boundary of Ω is of class C^3, there exists a pair $(\lambda, u) \in \mathbb{R}^+ \times C^d(\bar{\Omega}) := \mathbb{R}^+ \times C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\ \ \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

with u positive in Ω and $\max u \in (\pi, 3\pi/2)$ (see, for example, [3]). This is true in particular if Ω is the unit ball in $\mathbb{R}^N$. However, it is shown in [8] that if Ω is a hypercube there is no pair $(\lambda, u) \in \mathbb{R}^+ \times C^d(\bar{\Omega})$ satisfying (1.6), with u positive in Ω and $\max u \in (\pi, 3\pi/2)$. Since $3\pi/2$ is a supersolution of (1.6) for every $\lambda > 0$, there is no positive subsolution of (1.6) with maximum lying in $(\pi, 3\pi/2)$ for every $\lambda > 0$. For a proof of this statement and for more results in this direction, we refer the reader to [8] and [10].

2. PROOFS

PROOF OF THEOREM 1: Let u be a subsolution on Ω satisfying $\max_{\Omega} u > 0$. Without loss of generality we may assume, by using a translation, that the maximum of u is achieved at the origin. Since Ω is bounded, we may also assume, by using Tietze's theorem, that the function u is the restriction on $\bar{\Omega}$ of some continuous function on $\mathbb{R}^N$, nonpositive outside of Ω and zero outside of a ball large enough. We shall still denote the extended function by u. Define

$$u^*(r) = \max\{u(x);\ |x| = r\} \text{ for every } r \geq 0.$$

Observe that u^* is continuous since the following inequality holds:

$$|u^*(r_1) - u^*(r_2)| \leq \max_{|\theta|=1} |u(r_1\theta) - u(r_2\theta)| \tag{2.1}$$

and u is uniformly continuous on $\mathbb{R}^N$.

We also have

$$u^*(0) = u(0) = \max u = \max u^* > 0.$$

Denote by R the first zero of u^*. Define

$$v(x) = u^*(|x|) \text{ for } |x| \in [0,R].$$

Then v is continuous on $\bar{B}_R$, positive on B_R, $v = 0$ on ∂B_R, $\max_{B_R} v = \max_{\Omega} u$, and v is radially symmetric.

We shall prove that v is a subfunction on B_R, and therefore v satisfies all the required properties. By using partitions of unity and the compactness of the support of the test functions ϕ's, it is sufficient to prove that for every $x_0 \in B_R$ there is $r_0 > 0$ such that $B(x_0,r_0)$ lies in B_R and that v is a subfunction on $B(x_0,r_0)$. Let $x_0 \in B_R$ and $\alpha = v(x_0) > 0$. From the uniform continuity of u on $\mathbb{R}^N$, one finds $r_0 > 0$ such that

$$|u(x) - u(y)| < \frac{1}{3}\alpha \text{ for } |x-y| < r_0,\ x,y \in \mathbb{R}^N. \tag{2.2}$$

From (2.1), we also have

$$|u^*(s_1) - u^*(s_2)| < \frac{1}{3}\alpha \text{ for } |s_1-s_2| < r_0, s_1,s_2 \geq 0. \tag{2.3}$$

From (2.3) we get

$$u^*(r) > \frac{2}{2}\alpha > 0 \text{ for } |r - |x_0|| < r_0. \tag{2.4}$$

One finds that $B(x_0,r_0)$ lies in B_R, since u^* is continuous and vanishes at R. Define

$$\Sigma = \{\theta \in \mathbb{R}^N;\ |\theta| = 1\} \text{ and}$$

$$\Sigma' = \{\theta \in \Sigma;\ |x_0|\theta \in \Omega \text{ and } d(|x_0|\theta,\partial\Omega) \geq r_0\}. \tag{2.5}$$

Let $\theta \in \Sigma\setminus\Sigma'$ and $y \in B(|x_0|\theta,r_0)$, then it follows from (2.2) and the fact that $u \leq 0$ outside of Ω, that $u(y) < \frac{2}{3}\alpha$.

Recalling (2.4) we obtain

$$u^*(r) = \max\{u(r\theta);\ \theta \in \Sigma\} = \max\{u(r\theta);\ \theta \in \Sigma'\} \text{ for } |r-|x_0|| < r_0.$$

We can even choose a countable dense subset Σ'' of Σ' such that

$$u^*(r) = \max\{u(r\theta);\ \theta \in \Sigma''\} \text{ for } |r-|x_0|| < r_0.$$

Denote by $\theta_0,\theta_1,\theta_2,\ldots,\theta_n,\ldots$ the elements of Σ''. It follows from the definition of Σ', (2.5), that $B(|x_0|\theta_n,r_0) \subset \Omega$ for all $n \in \mathbb{N}$. Moreover, u is a subfunction on $B(|x_0|\theta_n,r_0)$ $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, there is a rotation R_n which maps $B(x_0,r_0)$ onto $B(|x_0|\theta_n,r_0)$ and such that

$$v(x) = \max\{u(R_n x);\ n \in \mathbb{N}\},\ x \in B(x_0,r_0).$$

From the rotation invariance in (1.2), it follows that the functions w_n defined by $w_n(x) = u(R_n x)$ are subfunctions on $B(x_0,r_0)$. Then

$$v_n = \max\{w_k;\ 0 \leqq k \leqq n\}$$

is an increasing sequence of subfunctions on $B(x_0,r_0)$ and by Dini's theorem $v = \sup_{n\geq 0} v_n = \lim_{n\to\infty} v_n$ is a subfunction on $B(x_0,r_0)$. This completes the proof of Theorem 1. □

PROOF OF THEOREM 2: Let u be a subsolution of Ω with $\rho = \max u > 0$. Let $R > 0$ and let v be the subsolution on B_R from Theorem 1.

Suppose that there is $s \in [0,\rho)$ such that $\int_s^\rho f(t)\,dt \leqq 0$. We will obtain a contradiction. For $n = 1,2,\ldots,$ define

$$f_n(t) = \begin{cases} f(t) & \text{for } t \leqq \rho, \\ f(\rho)[1+n(\rho-t)] & \text{for } t > \rho . \end{cases}$$

Then $f_n \in C(\mathbb{R})$ and $f_n(\rho + 1/n) = 0$. Then v is a radially symmetric subsolution to the problem

$$(P_n) \qquad \begin{cases} -\Delta u = f_n(u) & \text{on } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

and $\rho + 1/n$ is a radially symmetric supersolution of (P_n) satisfying $v \leqq \rho + 1/n$. By using a result of [4], there exists a solution u_n of (P_n), satisfying $v \leqq u_n \leqq \rho + 1/n$. By using a slight modification of the argument of [4], namely by applying the Schauder fixed-point theorem on the space of continuous functions on $\bar{B}_R$ which are radially symmetric, one can assume that u_n is radially symmetric. Then we obtain, since $u_n \in C^2(\bar{B}_R)$,

$$-u_n''(r) - \frac{N-1}{r} u_n'(r) = f_n(u_n(r)), \quad r \in [0,R], \qquad u_n'(r) = 0, \qquad u_n(R) = 0. \tag{2.6}$$

By integration, we obtain

$$\int_{u_n(r)}^{u_n(0)} f_n(s)\, ds \geq (N-1) \int_0^r \frac{1}{s} u_n'^2(s)\, ds, \; r \in [0,R]. \tag{2.7}$$

Since the functions $f_n(u_n)$ are uniformly bounded on $[0,R]$, it follows from (2.6) that the functions u_n, u_n' and u_n'' are uniformly bounded on $[0,R]$. For every $n \in \mathbb{N}$, there exists $r_n \in [0,R]$ such that $u_n(r_n) = s$. There exist a function $\bar{u} \in C^1[0,R]$, $\bar{r} \in [0,R]$ and a subsequence which we still denote by (u_n, r_n) such that u_n converges to $\bar{u}$ in $C^1[0,R]$ and r_n to $\bar{r}$.

Since $\rho > s = \lim_{n\to\infty} u_n(r_n) = \bar{u}(\bar{r})$ and $\bar{u}(0) = \rho$, we have $\bar{r} > 0$. From (2.7), we obtain

$$\int_0^{\bar{r}} \frac{1}{t} \bar{u}'^2(t)\, dt = \lim_{n\to\infty} \int_0^{r_n} \frac{1}{t} u_n'^2(t)\, dt = 0.$$

Hence $\bar{u}(t) = \rho$ on $[0,\bar{r}]$, and $s = \bar{u}(\bar{r}) = \rho$, a contradiction. This completes the proof of Theorem 2. □

REFERENCES

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] Ph. Clément and G. Sweers, Existence et multiplicité des solutions d'un problème aux valeurs propres elliptique semilinéaire, C.R. Acad. Sci. Paris 302, Série I, 19 (1986), 681-683.

[3] Ph. Clément and G. Sweers, Existence and multiplicity results for a semilinear elliptic eigenvalue problem, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. 13 (1987), 97-121.

[4] Ph. Clément and G. Sweers, Getting a solution between sub- and supersolutions without monotone iteration, Rend. dell'Instituto di Matematica dell'Università di Trieste, Vol. XIX (1987), 189-194.

[5] E.N. Dancer and K. Schmitt, On positive solutions of semilinear elliptic equations, Proc. A.M.S. 101 (1987), 445-452.

[6] D.G. De Figueiredo, On the uniqueness of positive solutions of the Dirichlet problem $-\Delta u = \lambda \sin u$, Nonlinear P.D.E. Appl. Collège de France Seminar, Vol. 7, Pitman, 1985, pp. 80-83.

[7] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209-243.

[8] G. Sweers, Doctoral Thesis, Delft, 1988.

[9] G. Sweers, On the maximum of solutions for a semilinear elliptic problem, Proceedings of the Royal Soc. of Edinburgh, 108A (1988), 357-370.

[10] G. Sweers, Semilinear elliptic problems on domains with corners, to appear in Commun. Partial Differ. Eq.

Ph. Clement and G. Sweers
Faculty of Mathematics
Delft University of Technology
Julianalaan 132
2628 BL Delft
THE NETHERLANDS