CORSO ESTIVO DI MATEMATICA
Differential Equations of Mathematical Physics

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It is better to have failed and tried,
To kick the groom and kiss the bride,
Than not to try and stand aside,
Sparing the coal as well as the guide.
John O'Mill

Contents

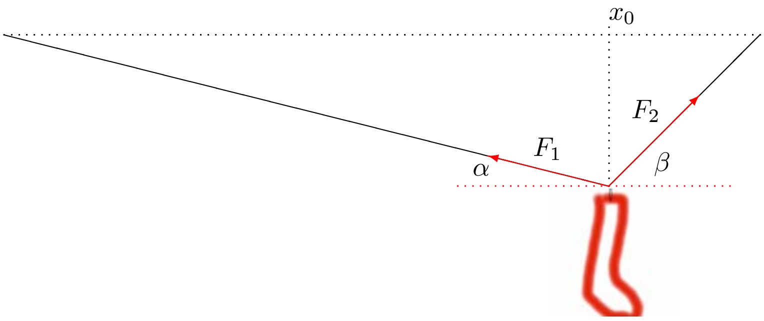
Week 1

From models to differential equations

1.1 Laundry on a line

1.1.1 A linear model

Consider a rope tied between two positions on the same height and hang a sock on that line at location x_0.



Then the balance of forces gives the following two equations:

$$F_1 \cos\alpha = F_2 \cos\beta = c,$$
$$F_1 \sin\alpha + F_2 \sin\beta = mg.$$

We assume that the positive constant c does not depend on the weight hanging on the line but is a given fixed quantity. The g is the gravitation constant and m the mass of the sock. Eliminating F_i we find

$$\tan\alpha + \tan\beta = \frac{mg}{c}.$$

We call u the deviation of the horizontal measured upwards so in the present situation u will be negative. Fixing the ends at $(0,0)$ and $(1,0)$ we find two straight lines connecting $(0,0)$ through $(x_0, u(x_0))$ to $(1,0)$. Moreover $u'(x) = \tan\beta$ for $x > x_0$ and $u'(x) = -\tan\alpha$ for $x < x_0$, so

$$u'(x_0^-) = -\tan\alpha \text{ and } u'(x_0^+) = \tan\beta.$$

This leads to
$$u'(x_0^+) - u'(x_0^-) = \frac{mg}{c}, \tag{1.1}$$
and the function u can be written as follows:
$$u(x) = \begin{cases} -\frac{mg}{c} x(1-x_0) & \text{for } x \le x_0, \\ -\frac{mg}{c} x_0(1-x) & \text{for } x > x_0. \end{cases} \tag{1.2}$$

Hanging multiple socks on that line, say at position x_i a sock of weight m_i with $i = 1 \ldots 35$, we find the function u by superposition. Fixing
$$G(x,s) = \begin{cases} x(1-s) & \text{for } x \le s, \\ s(1-x) & \text{for } x > s, \end{cases} \tag{1.3}$$
we'll get to
$$u(x) = \sum_{i=1}^{35} -\frac{m_i g}{c}\, G(x, x_i).$$
Indeed, in each point x_i we find
$$u'(x_i^+) - u'(x_i^-) = -\frac{m_i g}{c} \left[\frac{\partial}{\partial x} G(x, x_i) \right]_{x=x_i^-}^{x=x_i^+} = \frac{m_i g}{c}.$$

In the next step we will not only hang point-masses on the line but even a blanket. This gives a (continuously) distributed force down on this line. We will approximate this by dividing the line into n units of width $\triangle x$ and consider the force, say distributed with density $\rho(x)$, between $x_i - \frac{1}{2}\triangle x$ and $x_i + \frac{1}{2}\triangle x$ to be located at x_i. We could even allow some upwards pulling force and replace $-m_i g/c$ by $\rho(x_i)\triangle x$ to find
$$u(x) = \sum_{i=1}^{n} \rho(x_i) \triangle x\, G(x, x_i).$$
Letting $n \to \infty$ and this sum, a Riemann-sum, approximates an integral so that we obtain
$$u(x) = \int_0^1 G(x,s)\, \rho(s)\, ds. \tag{1.4}$$

We might also consider the balance of forces for the discretized problem and see that formula (1.1) gives
$$u'(x_i + \tfrac{1}{2}\triangle x) - u'(x_i - \tfrac{1}{2}\triangle x) = -\rho(x_i)\, \triangle x.$$
By Taylor
$$\begin{aligned} u'(x_i + \tfrac{1}{2}\triangle x) &= u'(x) + \tfrac{1}{2}\triangle x\ u''(x_i) + \mathcal{O}(\triangle x)^2, \\ u'(x_i - \tfrac{1}{2}\triangle x) &= u'(x) - \tfrac{1}{2}\triangle x\ u''(x_i) + \mathcal{O}(\triangle x)^2, \end{aligned}$$
and
$$\triangle x\ u''(x_i) + \mathcal{O}(\triangle x)^2 = -\rho(x_i)\, \triangle x.$$
After dividing by $\triangle x$ and taking limits this results in
$$-u''(x) = \rho(x). \tag{1.5}$$

Exercise 1 *Show that (1.4) solves (1.5) and satisfies the boundary conditions*

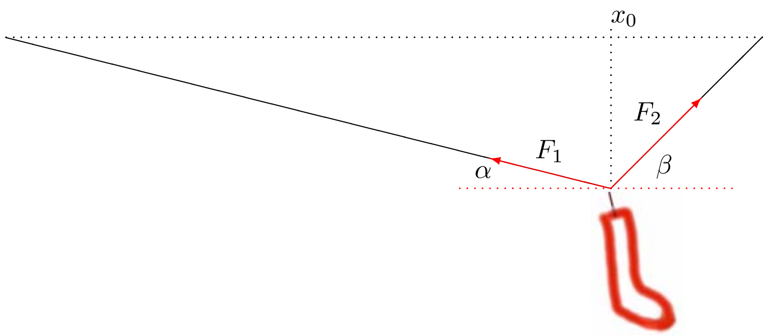
$$u(0) = u(1) = 0.$$

Definition 1.1.1 *The function in (1.3) is called a Green function for the boundary value problem*

$$\begin{cases} -u''(x) = \rho(x), \\ u(0) = u(1) = 0. \end{cases}$$

1.1.2 A nonlinear model

Consider again a rope tied between two positions on the same height and again hang a sock on that line at location x_0. Now we assume that the tension is constant throughout the rope.



To balance the forces we assume some sidewards effect of the wind. We find

$$\begin{gathered} F_1 = F_2 = c, \\ F_1 \sin\alpha + F_2 \sin\beta = mg. \end{gathered}$$

Now one has to use that

$$\sin\beta = \frac{u'\left(x_0^+\right)}{\sqrt{1+\left(u'\left(x_0^+\right)\right)^2}} \text{ and } \sin\alpha = \frac{-u'\left(x_0^-\right)}{\sqrt{1+\left(u'\left(x_0^-\right)\right)^2}}$$

and the replacement of (1.1) is

$$\frac{u'\left(x_0^+\right)}{\sqrt{1+\left(u'\left(x_0^+\right)\right)^2}} - \frac{u'\left(x_0^-\right)}{\sqrt{1+\left(u'\left(x_0^-\right)\right)^2}} = \frac{mg}{c}.$$

A formula as in (1.2) still holds but the superposition argument will fail since the problem is nonlinear. So no Green formula as before.

Nevertheless we may derive a differential equation as before. Proceeding as before the Taylor-expansion yields

$$-\frac{\partial}{\partial x}\left(\frac{u'(x)}{\sqrt{1+\left(u'(x)\right)^2}}\right) = \rho(x). \tag{1.6}$$

1.1.3 Comparing both models

The first derivation results in a linear equation which directly gives a solution formula. It is not that this formula is so pleasant but at least we find that whenever we can give some meaning to this formula (for example if $\rho \in L^1$) there exists a solution:

Lemma 1.1.2 *A function $u \in C^2[0,1]$ solves*

$$\begin{cases} -u''(x) = f(x) \text{ for } 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

if and only if

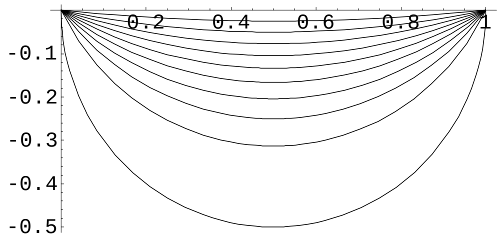
$$u(x) = \int_0^1 G(x,s)\, f(s) ds$$

with

$$G(x,s) = \begin{cases} x(1-s) & \text{for } 0 \le x \le s \le 1, \\ s(1-x) & \text{for } 0 \le s < x \le 1. \end{cases}$$

The second derivation results in a non-linear equation. We no longer see immediately if there exists a solution.

Exercise 2 *Suppose that we are considering equation (1.6) with a uniformly distributed force, say $\frac{g}{c}\rho(x) = M$. If possible compute the solution, that is, the function that satisfies (1.6) and $u(0) = u(1) = 0$. For which M does a solution exist? Hint: use that u is symmetric around $\frac{1}{2}$ and hence that $u'(\frac{1}{2}) = 0$.*



Here are some graphs of the solutions from the last exercise.

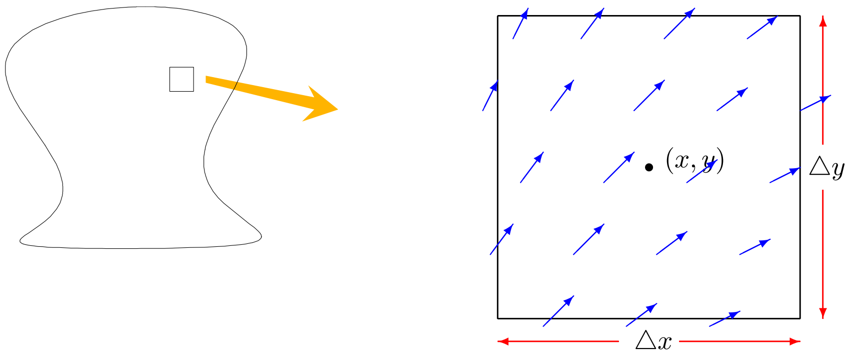
Remark 1.1.3 *If we don't have a separate weight hanging on the line but are considering a heavy rope that bends due to its own weight we find:*

$$\begin{cases} -u''(x) = c\sqrt{1 + (u'(x))^2} \text{ for } 0 < x < 1, \\ u(0) = u(1) = 0; \end{cases}$$

1.2 Flow through area and more 2d

Consider the flow through some domain Ω in $\mathbb{R}^2$ according to the velocity field $v = (v_1, v_2)$.

Definition 1.2.1 *A domain Ω is defined as an open and connected set. The boundary of the domain is called $\partial\Omega$ and its closure $\bar{\Omega} = \Omega \cup \partial\Omega$.*



If we single out a small rectangle with sides of length $\triangle x$ and $\triangle y$ with (x, y) in the middle we find

- flowing out:

$$\text{Out} = \int_{y-\frac{1}{2}\triangle y}^{y+\frac{1}{2}\triangle y} \rho . v_1 \left(x + \tfrac{1}{2}\triangle x, s\right) ds + \int_{x-\frac{1}{2}\triangle x}^{x+\frac{1}{2}\triangle x} \rho . v_2 \left(s, y + \tfrac{1}{2}\triangle y\right) ds,$$

- flowing in:

$$\text{In} = \int_{y-\frac{1}{2}\triangle y}^{y+\frac{1}{2}\triangle y} \rho . v_1 \left(x - \tfrac{1}{2}\triangle x, s\right) ds + \int_{x-\frac{1}{2}\triangle x}^{x+\frac{1}{2}\triangle x} \rho . v_2 \left(s, y - \tfrac{1}{2}\triangle y\right) ds.$$

Scaling with the size of the rectangle and assuming that v is sufficiently differentiable we obtain

$$\begin{aligned}
\lim_{\substack{\triangle y \downarrow 0 \\ \triangle x \downarrow 0}} \frac{\text{Out} - \text{In}}{\triangle x \triangle y} &= \lim_{\substack{\triangle y \downarrow 0 \\ \triangle x \downarrow 0}} \frac{\rho}{\triangle y} \int_{y-\frac{1}{2}\triangle y}^{y+\frac{1}{2}\triangle y} \frac{v_1 \left(x + \frac{1}{2}\triangle x, s\right) - v_1 \left(x - \frac{1}{2}\triangle x, s\right)}{\triangle x} ds + \\
&\quad + \lim_{\substack{\triangle y \downarrow 0 \\ \triangle x \downarrow 0}} \frac{\rho}{\triangle x} \int_{x-\frac{1}{2}\triangle x}^{x+\frac{1}{2}\triangle x} \frac{v_2 \left(s, y + \frac{1}{2}\triangle y\right) - v_2 \left(s, y - \frac{1}{2}\triangle y\right)}{\triangle y} ds
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\triangle y \downarrow 0} \frac{\rho}{\triangle y} \int_{y-\frac{1}{2}\triangle y}^{y+\frac{1}{2}\triangle y} \frac{\partial v_1}{\partial x} (x, s)\, ds + \lim_{\triangle x \downarrow 0} \frac{\rho}{\triangle x} \int_{x-\frac{1}{2}\triangle x}^{x+\frac{1}{2}\triangle x} \frac{\partial v_2}{\partial y} (s, y)\, ds \\
&= \rho \frac{\partial v_1 (x, y)}{\partial x} + \rho \frac{\partial v_2 (x, y)}{\partial y}.
\end{aligned}$$

If there is no concentration of fluid then the difference of these two quantities should be 0, that is $\nabla . v = 0$. In case of a potential flow we have $v = \nabla u$ for some function u and hence we obtain a harmonic function u:

$$\Delta u = \nabla . \nabla u = \nabla . v = 0.$$

A typical problem would be to determine the flow given some boundary data, for example obtained by measurements:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial n} u = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

The specific problem in (1.7) by the way is only solvable under the compatibility condition that the amount flowing in equals the amount going out:

$$\int_{\partial\Omega} \psi d\sigma = 0.$$

This physical condition does also follow 'mathematically' from (1.7). Indeed, by Green's Theorem:

$$\int_{\partial\Omega} \psi d\sigma = \int_{\partial\Omega} \frac{\partial}{\partial n} u d\sigma = \int_{\partial\Omega} n \cdot \nabla u \; d\sigma = \int_{\Omega} \nabla \cdot \nabla u \; dA = \int_{\Omega} \Delta u \; dA = 0.$$

In case that the potential is given at part of the boundary Γ and no flow in or out at $\partial\Omega \backslash \Gamma$ we find the problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \Gamma. \\ \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega \backslash \Gamma. \end{cases} \tag{1.8}$$

Such a problem could include local sources, think of external injection or extraction of fluid, and that would lead to $\Delta u = f$.

Instead of a rope or string showing a deviation due to some force applied one may consider a two-dimensional membrane. Similarly one may derive the following boundary value problems on a two-dimensional domain Ω.

1. For small forces/deviations we may consider the linear model as a reasonable model:
$$\begin{cases} -\Delta u(x, y) = f(x, y) \text{ for } (x, y) \in \Omega, \\ u(x, y) = 0 \text{ for } (x, y) \in \partial\Omega. \end{cases}$$
The differential operator Δ is called the Laplacian and is defined by $\Delta u = u_{xx} + u_{yy}$.

2. For larger deviations and assuming the tension is uniform throughout the domain the non-linear model:
$$\begin{cases} -\nabla \cdot \left(\dfrac{\nabla u(x, y)}{\sqrt{1 + |\nabla u(x, y)|^2}} \right) = f(x, y) \text{ for } (x, y) \in \Omega, \\ u(x, y) = 0 \text{ for } (x, y) \in \partial\Omega. \end{cases}$$
Here ∇ defined by $\nabla u = (u_x, u_y)$ is the gradient and $\nabla \cdot$ the divergence defined by $\nabla \cdot (v, w) = v_x + w_y$. For example the deviation u of a soap film with a pressure f applied is modeled by this boundary value problem.

3. In case that we do not consider a membrane that is isotropic (meaning uniform behavior in all directions), for example a stretched piece of cloth and the tension is distributed in the direction of the threads, the following nonlinear model might be better suited:

$$\begin{cases} -\frac{\partial}{\partial x}\left(\frac{u_x(x,y)}{\sqrt{1+(u_x(x,y))^2}}\right) - \frac{\partial}{\partial y}\left(\frac{u_y(x,y)}{\sqrt{1+(u_y(x,y))^2}}\right) = f(x,y) \text{ for } (x,y) \in \Omega, \\ u(x,y) = 0 \text{ for } (x,y) \in \partial\Omega. \end{cases} \tag{1.9}$$

In all of the above cases we have considered so-called zero Dirichlet boundary conditions. Instead one could also force the deviation u at the boundary by describing non-zero values.

Exercise 3 *Consider the second problem above with a constant right hand side on* $\Omega = \left\{(x,y)\,; x^2+y^2<1\right\}$:

$$\begin{cases} -\nabla\cdot\left(\dfrac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}}\right) = M \textit{ for } (x,y)\in\Omega, \\ u(x,y) = 0 \textit{ for } (x,y)\in\partial\Omega. \end{cases}$$

This would model a soap film attached on a ring with constant force (blowing onto the soap-film might give a good approximation). Assuming that the solution is radially symmetric compute the solution. Hint: in polar coordinates (r,φ) *with* $x = r\cos\varphi$ *and* $y = r\sin\varphi$ *we obtain by identifying* $U(r,\varphi) = u(r\cos\varphi, r\sin\varphi)$:

$$\begin{aligned} \frac{\partial}{\partial r}U &= \cos\varphi\frac{\partial}{\partial x}u + \sin\varphi\frac{\partial}{\partial y}u, \\ \frac{\partial}{\partial \varphi}U &= -r\sin\varphi\frac{\partial}{\partial x}u + r\cos\varphi\frac{\partial}{\partial y}u, \end{aligned}$$

and hence

$$\begin{aligned} \cos\varphi\frac{\partial}{\partial r}U - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}U &= \frac{\partial}{\partial x}u, \\ \sin\varphi\frac{\partial}{\partial r}U + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}U &= \frac{\partial}{\partial y}u. \end{aligned}$$

After some computations one should find for a radially symmetric function, that is $U = U(r)$, *that*

$$\nabla\cdot\left(\frac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}}\right) = \frac{\partial}{\partial r}\left(\frac{U_r}{\sqrt{1+U_r^2}}\right) + \frac{1}{r}\left(\frac{U_r}{\sqrt{1+U_r^2}}\right).$$

Exercise 4 *Next we consider a soap-film without any exterior force that is spanned between two rings of radius* 1 *and* 2 *with the middle one on level* $-C$ *for some positive constant* C *and the outer one at level* 0. *Mathematically:*

$$\begin{cases} -\nabla\cdot\left(\dfrac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}}\right) = 0 \textit{ for } 1<x^2+y^2<4, \\ u(x,y) = 0 \textit{ for } x^2+y^2=4, \\ u(x,y) = -C \textit{ for } x^2+y^2=1. \end{cases} \tag{1.10}$$

Solve this problem assuming that the solution is radial and C is small. What happens if C increases? Can one define a solution that is no longer a function? Hint: think of putting a virtual ring somewhere in between and think of two functions glued together appropriately.

Solution 4 *Writing* $F = \frac{U_r}{\sqrt{1+U_r^2}}$ *we have to solve* $F' + \frac{1}{r}F = 0$*. This is a separable o.d.e.:*

$$\frac{F'}{F} = -\frac{1}{r}$$

and an integration yields

$$\ln F = -\ln r + c_1$$

and hence $F(r) = \frac{c_2}{r}$ *for some* $c_2 > 0$*. Next solving* $\frac{U_r}{\sqrt{1+U_r^2}} = \frac{c_2}{r}$ *one finds*

$$U_r = \frac{c_2}{\sqrt{r^2 - c_2^2}}.$$

Hence

$$U(r) = c_2 \log\left(r + \sqrt{r^2 - c_2^2}\right) + c_3$$

The boundary conditions give

$$\begin{aligned} 0 &= U(2) = c_2 \log\left(2 + \sqrt{4 - c_2^2}\right) + c_3, \qquad (1.11) \\ -C &= U(1) = c_2 \log\left(1 + \sqrt{1 - c_2^2}\right) + c_3. \end{aligned}$$

Hence

$$c_2 \log\left(\frac{2 + \sqrt{4 - c_2^2}}{1 + \sqrt{1 - c_2^2}}\right) = C.$$

Note that $c_2 \mapsto c_2 \log\left(\frac{2+\sqrt{4-c_2^2}}{1+\sqrt{1-c_2^2}}\right)$ *is an increasing function that maps* $[0,1]$ *onto* $\left[0, \log\left(2+\sqrt{3}\right)\right]$*. Hence we find a function that solves the boundary value problem for* $C > 0$ *if and only if* $C \le \log\left(2+\sqrt{3}\right) \approx 1.31696$*. For those* C *there is a unique* $c_2 \in (0,1]$ *and with this* c_2 *we find* c_3 *by (1.11) and* U *:*

$$U(r) = c_2 \log\left(\frac{r + \sqrt{r^2 - c_2^2}}{2 + \sqrt{4 - c_2^2}}\right).$$

Let us consider more general solutions by putting in a virtual ring with radius R *on level* h*, necessarily* $R < 1$ *and* $-C < h < 0$*. To find two smoothly connected functions we are looking for a combination of the two solutions of*

$$\begin{cases} -\frac{\partial}{\partial r}\left(\frac{U_r}{\sqrt{1+U_r^2}}\right) + \left(\frac{U_r}{\sqrt{1+U_r^2}}\right) = 0 \text{ for } R < r < 2, \\ U(2) = 0 \text{ and } U(R) = h \text{ and } U_r(R) = \infty, \end{cases}$$

and

$$\begin{cases} -\frac{\partial}{\partial r}\left(\frac{\tilde{U}_r}{\sqrt{1+\tilde{U}_r^2}}\right) + \left(\frac{\tilde{U}_r}{\sqrt{1+\tilde{U}_r^2}}\right) = 0 \text{ for } R < r < 1, \\ \tilde{U}(1) = -C \text{ and } \tilde{U}(R) = h \text{ and } \tilde{U}_r(R) = -\infty. \end{cases}$$

Note that in order to connect smoothly we need a nonexistent derivative at R!

We proceed as follows. We know that

$$U(r) = c_2 \log\left(\frac{r+\sqrt{r^2-c_2^2}}{2+\sqrt{4-c_2^2}}\right)$$

is defined on $[c_2, 2]$ *and* $U_r(c_2) = \infty$. *Hence* $R = c_2$ *and*

$$U(R) = c_2 \log\left(\frac{c_2}{2+\sqrt{4-c_2^2}}\right).$$

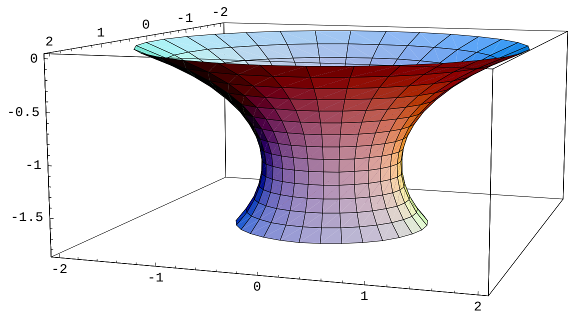
Then by symmetry

$$\tilde{U}(r) = 2U(R) - U(r)$$

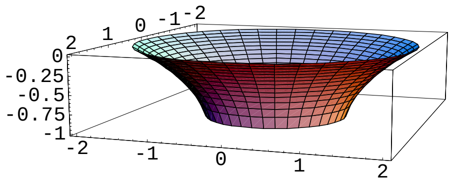
and next

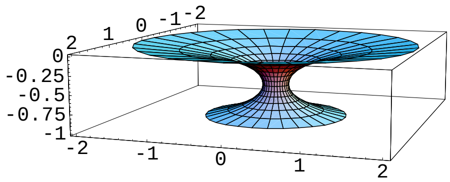
$$\begin{aligned}\tilde{U}(1) &= 2U(R) - U(1) = 2c_2 \log\left(\frac{c_2}{2+\sqrt{4-c_2^2}}\right) - c_2 \log\left(\frac{1+\sqrt{1-c_2^2}}{2+\sqrt{4-c_2^2}}\right) \\ &= c_2 \log\left(\frac{c_2^2}{\left(2+\sqrt{4-c_2^2}\right)\left(1+\sqrt{1-c_2^2}\right)}\right) = -C\end{aligned}$$

Again this function $c_2 \mapsto c_2 \log\left(\frac{c_2^2}{\left(2+\sqrt{4-c_2^2}\right)\left(1+\sqrt{1-c_2^2}\right)}\right)$ *is defined on* $[0,1]$. *The maximal* C *one finds is* ≈ 1.82617 *for* $c_2 \approx 0.72398$. *(Use Maple or Mathematica for this.)*



The solution that spans the maximal distance between the two rings.





Two solutions for the same configuration of rings.

Of the two solutions in the last figure only the left one is physically relevant. As a soap film the configuration on the right would immediately collapse. This lack of stability can be formulated in a mathematically sound way but for the moment we will skip that.

Exercise 5 *Solve the problem of the previous exercise for the linearized problem:*

$$\begin{cases} -\Delta u = 0 \text{ for } 1 < x^2 + y^2 < 4, \\ u(x,y) = 0 \text{ for } x^2 + y^2 = 4, \\ u(x,y) = -C \text{ for } x^2 + y^2 = 1. \end{cases}$$

Are there any critical C as for the previous problem? Hint: in polar coordinates one finds

$$\Delta = \left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial \varphi}\right)^2.$$

Better check this instead of just believing it!

Exercise 6 *Find a non-constant function u defined on $\bar{\Omega}$*

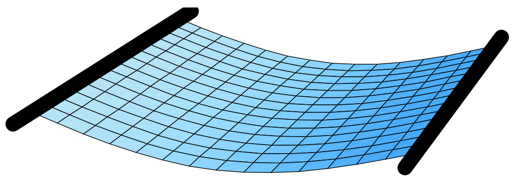
$$\Omega = \{(x,y); 0 < x < 1 \text{ and } 0 < y < 1\}$$

that satisfies

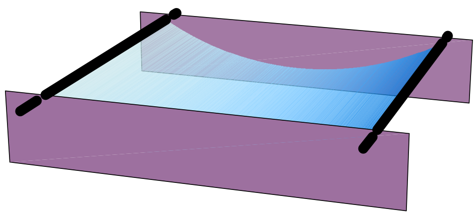
$$-\frac{\partial}{\partial x}\left(\frac{u_x(x,y)}{\sqrt{1+(u_x(x,y))^2}}\right) - \frac{\partial}{\partial y}\left(\frac{u_y(x,y)}{\sqrt{1+(u_y(x,y))^2}}\right) = 0 \text{ for } (x,y) \in \Omega, \tag{1.12}$$

the differential equation for a piece of cloth as in (1.9) now with $f = 0$. Give the boundary data $u(x,y) = g(x,y)$ for $(x,y) \in \partial\Omega$ that your solution satisfies.

Exercise 7 *Consider the linear model for a square blanket hanging between two straight rods when the deviation from the horizontal is due to some force f.*



Exercise 8 *Consider the non-linear model for a soap film with the following configuration:*



The film is free to move up and down the vertical front and back walls but is fixed to the rods on the right and left. There is a constant pressure from above. Give the boundary value problem and compute the solution.

1.3 Problems involving time

1.3.1 Wave equation

We have seen that a force due to tension in a string under some assumption is related to the second order derivative of the deviation from equilibrium. Instead of balancing this force by an exterior source such a 'force' can be neutralized by a change in the movement of the string. Even if we are considering time dependent deviations from equilibrium $u(x,t)$ the force-density due the tension in the (linearized) string is proportional to $u_{xx}(x,t)$. If this force (Newton's law: $F = ma$) implies a vertical movement we find

$$c_1 u_{xx}(x,t)\, dx = dF = dm\ u_{tt}(x,t) = \rho dx\ u_{tt}(x,t).$$

Here c depends on the tension in the string and ρ is the mass-density of that string. We find the 1-dimensional wave equation:

$$u_{tt} - c^2 u_{xx} = 0.$$

If we fix the string at its ends, say in 0 and ℓ and describe both the initial position and the initial velocity in all its points we arrive at the following system with $c, \ell > 0$ and ϕ and ψ given functions

$$\left\{\begin{array}{rll} \text{1d wave equation:} & u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{for } 0 < x < \ell \text{ and } t > 0, \\ \text{boundary condition:} & u(0,t) = u(\ell,t) = 0 & \text{for } t > 0, \\ \text{initial position:} & u(x,0) = \phi(x) & \text{for } 0 < x < \ell, \\ \text{initial velocity:} & u_t(x,0) = \psi(x) & \text{for } 0 < x < \ell. \end{array}\right.$$

One may show that this system is well-posed.

Definition 1.3.1 (Hadamard) *A system of differential equations with boundary and initial conditions is called well-posed if it satisfies the following properties:*

1. *It has a solution (existence).*
2. *There is at most one solution (uniqueness).*
3. *Small changes in the problem result in small changes in the solution (sensitivity).*

The last item is sometimes translated as '*the solution is continuously dependent on the boundary data*'.

We should remark that the second property may hold locally. Apart from this, these conditions stated as in this definition sound rather soft. For each problem we will study we have to specify in which sense these three properties hold.

Other systems that are well-posed (if we give the right specification of these three conditions) are the following.

Example 1.3.2 *For Ω a domain in $\mathbb{R}^2$ (think of the horizontal lay-out of a swimming pool of sufficient depth and u models the height of the surface of the water)*

$$\begin{cases} \textit{2d wave equation:} & u_{tt}(x,t) - c^2 \Delta u(x,t) = 0 & \textit{for } x \in \Omega \textit{ and } t > 0, \\ \textit{boundary condition:} & -\frac{\partial}{\partial n} u(x,t) + \alpha u\,(x,t) = 0 & \textit{for } x \in \partial\Omega \textit{ and } t > 0, \\ \textit{initial position:} & u(x,0) = \phi(x) & \textit{for } x \in \Omega, \\ \textit{initial velocity:} & u_t(x,0) = \psi(x) & \textit{for } x \in \Omega. \end{cases}$$

Here n denotes the outer normal, $\alpha > 0$ and $c > 0$ are some numbers and $\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2$.

Example 1.3.3 *For Ω a domain in $\mathbb{R}^3$ (think of room with students listening to some source of noise and u models the deviation of the pressure of the air compared with complete silence)*

$$\begin{cases} \begin{matrix}\textit{inhomogeneous} \\ \textit{3d wave equation:}\end{matrix} & u_{tt}(x,t) - c^2 \Delta u(x,t) = f\,(x,t) & \textit{for } x \in \Omega \textit{ and } t > 0, \\ \textit{boundary condition:} & -\frac{\partial}{\partial n} u(x,t) + \alpha(x) u\,(x,t) = 0 & \textit{for } x \in \partial\Omega \textit{ and } t > 0, \\ \textit{initial position:} & u(x,0) = 0 & \textit{for } x \in \Omega, \\ \textit{initial velocity:} & u_t(x,0) = 0 & \textit{for } x \in \Omega. \end{cases}$$

Here n denotes the outer normal and α a positive function defined on $\partial\Omega$. Again $c > 0$. On soft surfaces such as curtains α is large and on hard surfaces such as concrete α is small. The function f is given and usually almost zero except near some location close to the blackboard. The initial conditions being zero represents a teacher's dream.

Exercise 9 *Find all functions of the form $u\,(x,t) = \alpha(x)\beta(t)$ that satisfy*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \textit{for } 0 < x < \ell \textit{ and } t > 0, \\ u(0,t) = u(\ell,t) = 0 & \textit{for } t > 0, \\ u_t(x,0) = 0 & \textit{for } 0 < x < \ell. \end{cases}$$

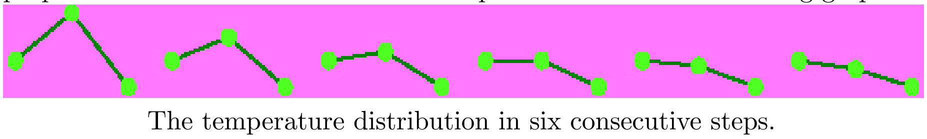
Exercise 10 *Same question for*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \textit{for } 0 < x < \ell \textit{ and } t > 0, \\ u(0,t) = u(\ell,t) = 0 & \textit{for } t > 0, \\ u(x,0) = 0 & \textit{for } 0 < x < \ell. \end{cases}$$

1.3.2 Heat equation

In the differential equation that models a temperature profile time also plays a role. It will appear in a different role as for the wave equation.

Allow us to give a simple explanation for the one-dimensional heat equation. If we consider a simple three-point model with the left and right point being kept at constant temperature we expect the middle point to reach the average temperature after due time. In fact one should expect the speed of convergence proportional to the difference with the equilibrium as in the following graph.



The temperature distribution in six consecutive steps.

For this discrete system with u_i for $i = 1, 2, 3$ the temperature of the middle node should satisfy:

$$\frac{\partial}{\partial t} u_2(t) = -c(-u_1 + 2u_2(t) - u_3).$$

If we would consider not just three nodes but say 43 of which we keep the first and the last one of fixed temperature we obtain

$$\begin{cases} u_1(t) = T_1, \\ \frac{\partial}{\partial t} u_i(t) = -c(-u_{i-1}(t) + 2u_i(t) - u_{i+1}(t)) \text{ for } i \in \{2, ..., 42\}, \\ u_{43}(t) = T_2. \end{cases}$$

Letting the stepsize between nodes go to zero, but adding more nodes to keep the total length fixed, and applying the right scaling: $u(i\triangle x, t) = u_i(t)$ and $c = \tilde{c}(\triangle x)^{-2}$ we find for smooth u through

$$\lim_{\triangle x \downarrow 0} \frac{-u(x - \triangle x, t) + 2u(x, t) - u(x + \triangle x, t)}{(\triangle x)^2} = -u_{xx}(x, t)$$

the differential equation

$$\frac{\partial}{\partial t} u(x, t) = \tilde{c}\, u_{xx}(x, t).$$

For $\tilde{c} > 0$ this is the 1-d heat equation.

Example 1.3.4 *Considering a cylinder of radius d and length ℓ which is isolated around with the bottom in ice and heated from above. One finds the following model which contains a 3d heat equation:*

$$\begin{cases} u_t(x,t) - c^2 \Delta u(x,t) = 0 & \text{for } x_1^2 + x_2^2 < d^2,\ 0 < x_3 < \ell \text{ and } t > 0, \\ \frac{\partial}{\partial n} u(x,t) = 0 & \text{for } x_1^2 + x_2^2 = d^2,\ 0 < x_3 < \ell \text{ and } t > 0, \\ u(x,t) = 0 & \text{for } x_1^2 + x_2^2 < d^2,\ x_3 = 0 \text{ and } t > 0, \\ u(x,t) = \phi(x_1, x_2) & \text{for } x_1^2 + x_2^2 < d^2,\ x_3 = \ell \text{ and } t > 0, \\ u(x,0) = 0 & \text{for } x_1^2 + x_2^2 < d^2,\ 0 < x_3 < \ell. \end{cases}$$

Here $\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2$ only contains the derivatives with respect to space.

Until now we have seen several types of boundary conditions. Let us give the names that belong to them.

Definition 1.3.5 *For second order ordinary and partial differential equations on a domain Ω the following names are given:*

- *Dirichlet:* $u(x) = \phi(x)$ *on* $\partial\Omega$ *for a given function* ϕ.
- *Neumann:* $\frac{\partial}{\partial n} u(x) = \psi(x)$ *on* $\partial\Omega$ *for a given function* ψ.
- *Robin:* $\alpha(x) \frac{\partial}{\partial n} u(x) = u(x) + \chi(x)$ *on* $\partial\Omega$ *for a given function* χ.

Here n is the outward normal at the boundary.

Exercise 11 *Consider the 1d heat equation for $(x,t) \in (0,\ell) \times \mathbb{R}^+$:*

$$u_t = c^2 u_{xx}. \tag{1.13}$$

1. *Suppose $u\,(0,t) = u(\ell,t) = 0$ for all $t > 0$. Compute the only positive functions satisfying these boundary conditions and (1.13) of the form $u\,(x,t) = X(x)T(t)$.*

2. *Suppose $\frac{\partial}{\partial n} u\,(0,t) = \frac{\partial}{\partial n} u(\ell,t) = 0$ for all $t > 0$. Same question.*

3. *Suppose $\alpha \frac{\partial}{\partial n} u\,(0,t) + u\,(0,t) = \frac{\partial}{\partial n} u(\ell,t) + u\,(0,t) = 0$ for all $t > 0$. Same question again. Thinking of physics: what would be a restriction for the number α?*

1.4 Differential equations from calculus of variations

A serious course with the title Calculus of Variations would certainly take more time then just a few hours. Here we will just borrow some techniques in order to derive some systems of differential equations.

Instead of trying to find some balance of forces in order to derive a differential equation and the appropriate boundary conditions one can try to minimize for example the energy of the system. Going back to the laundry on a line from $(0,0)$ to $(1,0)$ one could think of giving a formula for the energy of the system. First the energy due to deviation of the equilibrium:

$$E_{rope}\left(y\right) = \int_0^\ell \tfrac{1}{2}(y'(x))^2 dx$$

Also the potential energy due to the laundry pulling the rope down with force f is present:

$$E_{pot}\left(y\right) = -\int_0^\ell y\left(x\right) f(x) dx.$$

So the total energy is

$$E\left(y\right) = \int_0^\ell \left(\tfrac{1}{2}(y'(x))^2 - y\left(x\right) f(x)\right) dx. \tag{1.14}$$

For the line with constant tension one has

$$E\left(y\right) = \int_0^\ell \left(\sqrt{1+(y'(x))^2} - 1 - y\left(x\right) f(x)\right) dx. \tag{1.15}$$

Assuming that a solution minimizes the energy implies that for any nice function η and number τ it should hold that

$$E\left(y\right) \leq E\left(y+\tau\eta\right).$$

Nice means that the boundary condition also hold for the function $y+\tau\eta$ and that η has some differentiability properties.

If the functional $\tau \mapsto E\left(y+\tau\eta\right)$ is differentiable, then y should be a stationary point:

$$\frac{\partial}{\partial\tau} E\left(y+\tau\eta\right)_{\tau=0} = 0 \text{ for all such } \eta.$$

Definition 1.4.1 *The functional $J_1\left(y;\eta\right) := \frac{\partial}{\partial\tau} E\left(y+\tau\eta\right)_{\tau=0}$ is called the first variation of E.*

Example 1.4.2 *We find for the E in (1.15) that*

$$\frac{\partial}{\partial\tau} E\left(y+\tau\eta\right)_{\tau=0} = \int_0^\ell \left(\frac{y'(x)\eta'(x)}{\sqrt{1+(y'(x))^2}} - \eta\left(x\right) f(x)\right) dx. \tag{1.16}$$

If the function y is as we want then this quantity should be 0 for all such η. With an integration by part we find

$$0 = \left[\frac{y'(x)}{\sqrt{1+(y'(x))^2}}\eta(x)\right]_0^\ell - \int_0^\ell \left(\left(\frac{y'(x)}{\sqrt{1+(y'(x))^2}}\right)' + f(x)\right) \eta\left(x\right) dx.$$

Since y and $y+\tau\eta$ are zero in 0 and ℓ the boundary terms disappear and we are left with

$$\int_0^\ell \left(\left(\frac{y'(x)}{\sqrt{1+(y'(x))^2}} \right)' + f(x) \right) \eta(x)\, dx = 0.$$

If this holds for all functions η then

$$-\left(\frac{y'(x)}{\sqrt{1+(y'(x))^2}} \right)' = f(x).$$

Here we have our familiar differential equation.

Definition 1.4.3 *The differential equation that follows for y from*

$$\frac{\partial}{\partial\tau} E\left(y+\tau\eta\right)_{\tau=0} = 0 \text{ for all appropriate } \eta, \tag{1.17}$$

is called the Euler-Lagrange equation for E.
The extra boundary conditions for y that follow from (1.17) are called the natural boundary conditions.

Example 1.4.4 *Let us consider the minimal surface that connects a ring of radius 2 on level 0 with a ring of radius 1 on level -1. Assuming the solution is a function defined between these two rings, $\Omega = \left\{(x,y); 1 < x^2+y^2 < 4\right\}$ is the domain. If u is the height of the surface then the area is*

$$Area\,(u) = \int_\Omega \sqrt{1+\left|\nabla u(x,y)\right|^2} dxdy.$$

We find

$$\frac{\partial}{\partial\tau} Area\left(y+\tau\eta\right)_{\tau=0} = \int_\Omega \frac{\nabla u(x,y)\cdot\nabla\eta(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} dxdy. \tag{1.18}$$

Since u and $u+\tau\eta$ are both equal to 0 on the outer ring and equal to -1 on the inner ring it follows that $\eta=0$ on both parts of the boundary. Then by Green

$$\begin{aligned} 0 &= \int_\Omega \frac{\nabla u(x,y)\cdot\nabla\eta(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} dxdy = \\ &= \int_{\partial\Omega} \frac{\frac{\partial}{\partial n} u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} \eta(x,y) d\sigma - \int_\Omega \left(\nabla\cdot \frac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} \right) \eta(x,y) dxdy \\ &= -\int_\Omega \left(\nabla\cdot \frac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} \right) \eta(x,y) dxdy. \end{aligned}$$

If this holds for all functions η then a solution u satisfies

$$\nabla\cdot \frac{\nabla u(x,y)}{\sqrt{1+\left|\nabla u(x,y)\right|^2}} = 0.$$

Example 1.4.5 *Consider a one-dimensional loaded beam of shape $y(x)$ that is fixed at its boundaries by $y(0) = y(\ell) = 0$. The energy consists of the deformation energy:*

$$E_{def}(y) = \int_0^\ell \frac{1}{2} \left(y''(x)\right)^2 dx$$

and the potential energy due to the weight that pushes:

$$E_{pot}(y) = \int_0^\ell y(x) f(x) dx.$$

So the total energy $E = E_{def} - E_{pot}$ equals

$$E(y) = \int_0^\ell \left(\frac{1}{2} \left(y''(x)\right)^2 - y(x) f(x) \right) dx$$

and its first variation

$$\frac{\partial}{\partial \tau} E\left(y + \tau\eta\right)_{\tau=0} = \int_0^\ell y''(x)\eta''(x) - \eta(x) f(x) \ dx =$$

$$= \Big[y''(x)\eta'(x) - y'''(x)\eta(x) \Big]_0^\ell + \int_0^\ell y''''(x)\eta(x) - \eta(x) f(x) \ dx. \tag{1.19}$$

Since y and $y + \tau\eta$ equals 0 on the boundary we find that η equals 0 on the boundary. Using this only part of the boundary terms disappear:

$$(\#1.19) = \Big[y''(x)\eta'(x) \Big]_0^\ell + \int_0^\ell \left(y''''(x) - f(x)\right) \eta(x) \ dx.$$

If this holds for all functions η with $\eta = 0$ at the boundary then we may first consider functions which also disappear in the first derivative at the boundary and find that $y''''(x) = f(x)$ for all $x \in (0, \ell)$. Next by choosing η with $\eta' \neq 0$ at the boundary we additionally find that a solution y satisfies

$$y''(0) = y''(\ell) = 0.$$

So the solution should satisfy:

$$\begin{cases} \text{the d.e.:} & y''''(x) = f(x), \\ \text{prescribed b.c.:} & y(0) = y(\ell) = 0, \\ \text{natural b.c.:} & y''(0) = y''(\ell) = 0. \end{cases}$$

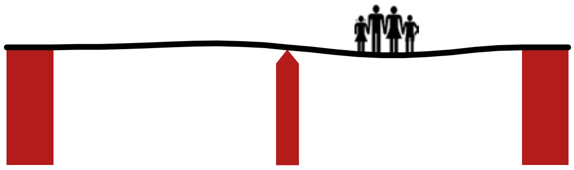
Exercise 12 *At most swimming pools there is a possibility to jump in from a springboard. Just standing on it the energy is, supposing it is a 1-d model, as for the loaded beam but with different boundary conditions, namely $y(0) = y'(0) = 0$ and no prescribed boundary condition at ℓ. Give the corresponding boundary value problem.*

Consider the problem

$$\begin{cases} y''''(x) = f(x), \text{ for } 0 < x < 1 \\ y(0) = y'(0) = 0, \\ y''(1) = y'''(1) = 0. \end{cases} \tag{1.20}$$

Exercise 13 *Show that the problem (1.20) is well-posed (specify your setting). The solution of (1.20) can be written by means of a Green function:* $u(x) = \int_0^1 G(x,s) f(s) ds$. *Compute this Green function.*

Exercise 14 *Consider a bridge that is clamped in* -1 *and* 1 *and which is supported in* 0 *(and we assume it doesn't loose contact which the supporting pillar in* 0*).*



If u *is the deviation the prescribed boundary conditions are:*

$$u(-1) = u'(-1) = u(0) = u(1) = u'(1) = 0.$$

The energy functional is

$$E(u) = \int_{-1}^{1} \left(\frac{1}{2} (u''(x))^2 - f(x)\ u(x) \right) dx.$$

Compute the differential equation for the minimizing solution and the remaining natural boundary conditions that this function should satisfy. Hint: consider u_ℓ *on* $[-1,0]$ *and* u_r *on* $[0,1]$. *How many conditions are needed?*

1.5 Mathematical solutions are not always physically relevant

If we believe physics in the sense that for physically relevant solutions the energy is minimized we could come back to the two solutions of the soap film between the two rings as in the last part of Exercise 4. We now do have an energy for that model, at least when we are discussing functions u of (x, y) :

$$E(u) = \int_\Omega \sqrt{1 + |\nabla u(x,y)|^2} dxdy.$$

We have seen that if u is a solution then the first variation necessarily is 0 for all appropriate test functions η :

$$\frac{\partial}{\partial \tau} E(u + \tau\eta)_{\tau=0} = 0.$$

For twice differentiable functions Taylor gives $f(a + \tau b) = f(a) + \tau f'(a) + \frac{1}{2}\tau^2 f''(a) + o(\tau^2)$. So in order that the energy-functional has a minimum it will be necessary that

$$\left(\frac{\partial}{\partial \tau}\right)^2 E(u + \tau\eta)_{\tau=0} \geq 0$$

whenever E is a 'nice' functional.

Definition 1.5.1 *The quantity* $\left(\frac{\partial}{\partial \tau}\right)^2 E(u + \tau\eta)_{\tau=0}$ *is called the second variation.*

Example 1.5.2 *Back to the rings connected by the soap film from Exercise 4. We were looking for radially symmetric solutions of (1.10). If we restrict ourselves to solutions that are functions of r then we found*

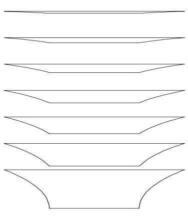
$$U(r) = c_2 \log\left(\frac{r + \sqrt{r^2 - c_2^2}}{2 + \sqrt{4 - c_2^2}}\right) \tag{1.21}$$

if and only if $C \leq \log\left(2 + \sqrt{3}\right) \approx 1.31696$. Let call these solutions of type I.

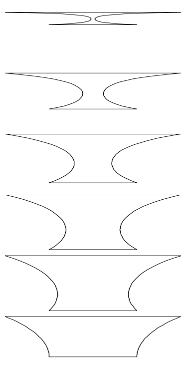
If we allowed graphs that are not necessarily represented by one function then there are two solutions if and only if $C \leq C_{\max} \approx 1.82617$:

$$U(r) = c_2 \log\left(\frac{r+\sqrt{r^2-c_2^2}}{2+\sqrt{4-c_2^2}}\right) \text{ and } \tilde{U}(r) = c_2 \log\left(\frac{c_2^2}{\left(2+\sqrt{4-c_2^2}\right)\left(r+\sqrt{r^2-c_2^2}\right)}\right).$$

Let us call these solutions of type II.



type I



type II

On the left are type I and on the right type II solutions. The top two or three on the right probably do not exist 'physically'.

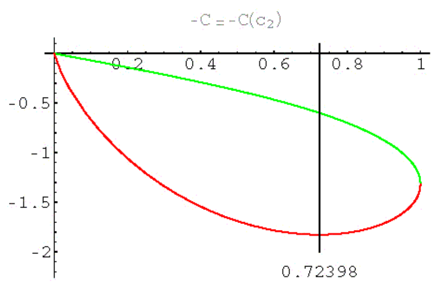
The energy of the type I solution U in (1.21) is (skipping the constant 2π)

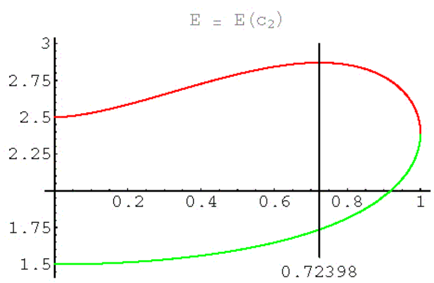
$$\begin{aligned}
\int_\Omega \sqrt{1+|\nabla U|^2}dxdy &= \int_1^2 \sqrt{1+U_r^2(r)}rdr \\
&= \int_1^2 \sqrt{1+\left(\frac{c_2}{\sqrt{r^2-c_2^2}}\right)^2}\ r\ dr \\
&= \int_1^2 \frac{r^2}{\sqrt{r^2-c_2^2}}dr = \\
&= \left[\frac{r\sqrt{r^2-c_2^2}}{2}+\frac{c_2^2\log(r+\sqrt{r^2-c_2^2})}{2}\right]_1^2 \\
&= \sqrt{4-c_2^2}-\tfrac{1}{2}\sqrt{1-c_2^2}+\tfrac{1}{2}c_2^2\log\left(\frac{2+\sqrt{4-c_2^2}}{1+\sqrt{1-c_2^2}}\right).
\end{aligned}$$

The energy of type II solution equals

$$\begin{aligned}
&\int_{c_2}^2 \sqrt{1+U_r^2(r)}rdr+\int_{c_2}^1\sqrt{1+\tilde{U}_r^2(r)}rdr = \\
&= \int_1^2 \sqrt{1+U_r^2(r)}rdr+2\int_{c_2}^1\sqrt{1+U_r^2(r)}rdr \\
&= \sqrt{4-c_2^2}-\tfrac{1}{2}\sqrt{1-c_2^2}+\tfrac{1}{2}c_2^2\log\left(\frac{2+\sqrt{4-c_2^2}}{1+\sqrt{1-c_2^2}}\right)+ \\
&\qquad +\sqrt{1-c_2^2}+c_2^2\log(1+\sqrt{1-c_2^2})-c_2^2\log(c_2) \\
&= \sqrt{4-c_2^2}+\tfrac{1}{2}\sqrt{1-c_2^2}+\tfrac{1}{2}c_2^2\log\left(\frac{\left(2+\sqrt{4-c_2^2}\right)\left(1+\sqrt{1-c_2^2}\right)}{c_2^2}\right).
\end{aligned}$$

Just like $C=C(c_2)$ also $E=E(c_2)$ turns out to be an ugly function. But with the help of Mathematica (or Maple) we may plot these functions.





On the left: C_{I} above C_{II}, on the right: E_{I} below E_{II}.

Exercise 15 *Use these plots of height and energy as functions of c_2 to compare the energy of the two solutions for the same configuration of rings. Can one conclude instability for one of the solutions? Are physically relevant solutions necessarily global minimizers or could they be local minimizers of the energy functional? Hint: think of the surface area of these solutions.*

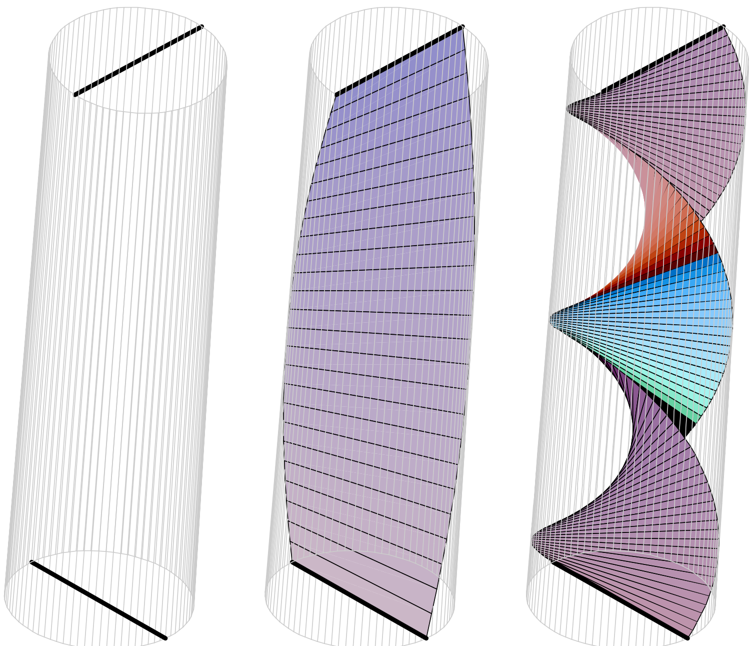
Exercise 16 *The approach that uses solutions in the form $r \mapsto U(r)$ in Exercise 4 has the drawback that we have to glue together two functions for one solution. A formulation without this problem uses $u \mapsto R(u)$ (the 'inverse' function). Give the energy $\tilde{E}$ formulation for this R. What is the first variation for this $\tilde{E}$?*

Using this new formulation it might be possible to find a connection from a type II solution R_{II} to a type I solution R_{I} with the same $C < \log\left(2+\sqrt{3}\right)$ such that the energy decreases along the connecting path. To be more precise: we should construct a homotopy $H \in C\left([0,1]\,; C^2\left[-C,0\right]\right)$ with $H(0) = R_{\mathrm{II}}$ and $H(1) = R_{\mathrm{I}}$ that is such that $t \mapsto \tilde{E}(H(t))$ is decreasing. This would prove that R_{II} is not even a local minimizer.

Claim 1.5.3 *A type II solution with $c_2 < 0.72398$ is probably not a local minimizer of the energy and hence not a physically relevant solution.*

Volunteers may try to prove this claim.

Exercise 17 *Let us consider a glass cylinder with at the top and bottom each a rod laid out on a diametral line and such that both rods cross perpendicular. See the left of the three figures:*



One can connect these two rods through this cylinder by a soap film as in the two figures on the right.

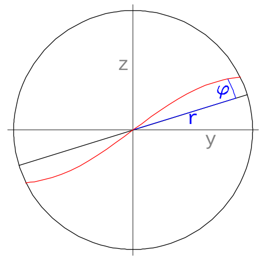
1. *The solution depicted in the second cylinder is described by a function. Setting the cylinder* $\left\{(x,y,z)\,;-\ell\leq x\leq\ell \text{ and } y^2+z^2=R^2\right\}$ *the function looks as* $z(x,y)=y\tan\left(\frac{\pi}{4}\frac{x}{\ell}\right)$. *Show that for a deviation of this function by* w *the area formula is given by*

$$E\left(w\right)=\int_{-\ell}^{\ell}\int_{y_{l,w}(x)}^{y_{r,w}(x)}\sqrt{1+\left|\nabla z+\nabla w\right|^2}dydx$$

where $y_{l,w}(x)<0<y_{r,w}(x)$ *are in absolute sense smallest solutions of*

$$y^2+\left(y\tan\left(\frac{\pi}{4}\frac{x}{\ell}\right)+w(x,y)\right)^2=R^2.$$

2. *If we restrict ourselves to functions that satisfy* $w(x,0)=0$ *we may use a more suitable coordinate system, namely* x,r *instead of* x,y *and describe* $\varphi(x,r)$ *instead of* $w(x,y)$ *as follows:*



Show that

$$E\left(\varphi\right)=\int_{-\ell}^{\ell}\int_{-R}^{R}\sqrt{1+\left(\frac{\partial\varphi}{\partial r}\right)^2+\left(r\frac{\pi}{4\ell}+\frac{\partial\varphi}{\partial x}\right)^2}drdx.$$

3. *Derive the Euler-Lagrange equation for* $E(\varphi)$ *and state the boundary value problem.*

4. *Show that 'the constant turning plane' is indeed a solution in the sense that it yields a stationary point of the energy functional.*

5. *It is not obvious to me if the first depicted solution is indeed a minimizer. If you ask me for a guess: for* $R>>\ell$ *it is the minimizer* ($R>1.535777884999\ \ell$*?). For the second depicted solution of the boundary value problem one can find a geometrical argument that shows it is not even a local minimizer. (Im)prove these guesses.*

Week 2

Spaces, Traces and Imbeddings

2.1 Function spaces

2.1.1 Hölder spaces

Before we will start 'solving problems' we have to fix the type of function we are looking for. For mathematics that means that we have to decide which function space will be used. The classical spaces that are often used for solutions of p.d.e. (and o.d.e.) are $C^k(\bar{\Omega})$ and the Hölder spaces $C^{k,\alpha}(\bar{\Omega})$ with $k \in \mathbb{N}$ and $\alpha \in (0,1]$. Here is a short reminder of those spaces.

Definition 2.1.1 *Let $\Omega \subset \mathbb{R}^n$ be a domain.*

- *$C(\bar{\Omega})$ with $\|\cdot\|_\infty$ defined by $\|u\|_\infty = \sup_{x\in\bar{\Omega}} |u(x)|$ is the space of continuous functions.*
- *Let $\alpha \in (0,1]$. Then $C^\alpha(\bar{\Omega})$ with $\|\cdot\|_\alpha$ defined by*

  $$\|u\|_\alpha = \|u\|_\infty + \sup_{x,y\in\bar{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

  consists of all functions $u \in C(\bar{\Omega})$ such that $\|u\|_\alpha < \infty$. It is the space of functions which are Hölder-continuous with coefficient α.

If we want higher order derivatives to be continuous up to the boundary we should explain how these are defined on the boundary or to consider just derivatives in the interior and extend these. We choose this second option.

Definition 2.1.2 *Let $\Omega \subset \mathbb{R}^n$ be a domain and $k \in \mathbb{N}^+$.*

- *$C^k(\bar{\Omega})$ with $\|\cdot\|_{C^k(\bar{\Omega})}$ consists of all functions u that are k-times differentiable in Ω and which are such that for each multiindex $\beta \in \mathbb{N}^n$ with $1 \le |\beta| \le k$ there exists $g_\beta \in C(\bar{\Omega})$ with $g_\beta = \left(\frac{\partial}{\partial x}\right)^\beta u$ in Ω. The norm $\|\cdot\|_{C^k(\bar{\Omega})}$ is defined by*

  $$\|u\|_{C^k(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + \sum_{1\le|\beta|\le k} \|g_\beta\|_{C(\bar{\Omega})}.$$

- $C^{k,\alpha}(\bar{\Omega})$ *with* $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ *consists of all functions* u *that are* k *times* α-*Hölder-continuously differentiable in* Ω *and which are such that for each multiindex* $\beta \in \mathbb{N}^n$ *with* $1 \leq |\beta| \leq k$ *there exists* $g_\beta \in C^\alpha(\bar{\Omega})$ *with* $g_\beta = \left(\frac{\partial}{\partial x}\right)^\beta u$ *in* Ω. *The norm* $\|\cdot\|_{C^{k,\alpha}(\bar{\Omega})}$ *is defined by*

$$\|u\|_{C^{k,\alpha}(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + \sum_{1 \leq |\beta| < k} \|g_\beta\|_{C(\bar{\Omega})} + \sum_{|\beta| = k} \|g_\beta\|_{C^\alpha(\bar{\Omega})} .$$

Remark 2.1.3 *Adding an index* 0 *at the bottom in* $C_0^{k,\alpha}(\bar{\Omega})$ *means that* u *and all of its derivatives (read* g_β*) up to order* k *are* 0 *at the boundary. For example,*

$$C^4(\bar{\Omega}) \cap C_0^2(\bar{\Omega}) = \left\{ u \in C^4(\bar{\Omega}); u_{|\partial\Omega} = \left(\frac{\partial}{\partial x_i} u\right)_{|\partial\Omega} = \left(\frac{\partial^2}{\partial x_i \partial x_j} u\right)_{|\partial\Omega} = 0 \right\}$$

is a subspace of $C^4(\bar{\Omega})$.

Remark 2.1.4 *Some other cases where we will use the notation* $C^{\cdot}(\cdot)$ *are the following. For those cases we are not defining a norm but just consider the collection of functions.*

The set $C^\infty(\bar{\Omega})$ *consists of the functions that belong to* $C^k(\bar{\Omega})$ *for every* $k \in \mathbb{N}$.

If we write $C^m(\Omega)$, *without the closure of* Ω, *we mean all functions which are* m-*times differentiable inside* Ω.

By $C_0^\infty(\Omega)$ *we will mean all functions with compact support in* Ω *and which are infinitely differentiable on* Ω.

For bounded domains Ω the spaces $C^k(\bar{\Omega})$ and $C^{k,\alpha}(\bar{\Omega})$ are Banach spaces.

Although often well equipped for solving differential equations these spaces are in general not very convenient in variational settings. Spaces that are better suited are the...

2.1.2 Sobolev spaces

Let us first recall that $L^p(\Omega)$ is the space of all measurable functions $u : \Omega \to \mathbb{R}$ with $\int_\Omega |u(x)|^p \, dx < \infty$.

Definition 2.1.5 *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and* $k \in \mathbb{N}$ *and* $p \in (1, \infty)$.

- $W^{k,p}(\Omega)$ *are the functions* u *such that* $\left(\frac{\partial}{\partial x}\right)^\alpha u \in L^p(\Omega)$ *for all* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \leq k$. *Its norm* $\|\cdot\|_{W^{k,p}(\Omega)}$ *is defined by*

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha u \right\|_{L^p(\Omega)} .$$

Remark 2.1.6 *When is a derivative in* $L^p(\Omega)$*? For the moment we just recall that* $\frac{\partial}{\partial x_1} u \in L^p(\Omega)$ *if there is* $g \in L^p(\Omega)$ *such that*

$$\int_\Omega g \, \varphi \, dx = -\int_\Omega u \, \tfrac{\partial}{\partial x_1} \varphi \, dx \text{ for all } \varphi \in C_0^\infty(\Omega).$$

For $u \in C^1(\Omega) \cap L^p(\Omega)$ one takes $g = \frac{\partial}{\partial x_1} u \in C(\Omega)$, checks if $g \in L^p(\Omega)$, and the formula follows from an integration by parts since each φ has a compact support in Ω.

A closely related space is used in case of Dirichlet boundary conditions. Then we would like to restrict ourselves to all functions $u \in W^{k,p}(\Omega)$ that satisfy $\left(\frac{\partial}{\partial x}\right)^\alpha u = 0$ on $\partial\Omega$ for α with $0 \leq |\alpha| \leq m$ for some m. Since the functions in $W^{k,p}(\Omega)$ are not necessarily continuous we cannot assume that a condition like $\left(\frac{\partial}{\partial x}\right)^\alpha u = 0$ on $\partial\Omega$ to hold pointwise. One way out is through the following definition.

Definition 2.1.7 *Let $\Omega \subset \mathbb{R}^n$ be a domain, $k \in \mathbb{N}$ and $p \in (1, \infty)$.*

- *Set $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{k,p}(\Omega)}}$, the closure in the $\|\cdot\|_{W^{k,p}(\Omega)}$-norm of the set of infinitely differentiable functions with compact support in Ω. The standard norm on $W_0^{k,p}(\Omega)$ is $\|\cdot\|_{W^{k,p}(\Omega)}$.*

Instead of the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ one might also encounter some other norm for $W_0^{k,p}(\Omega)$ namely $|||\cdot|||_{W_0^{k,p}(\Omega)}$ defined by taking only the highest order derivatives:

$$|||u|||_{W_0^{k,p}(\Omega)} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha u \right\|_{L^p(\Omega)},$$

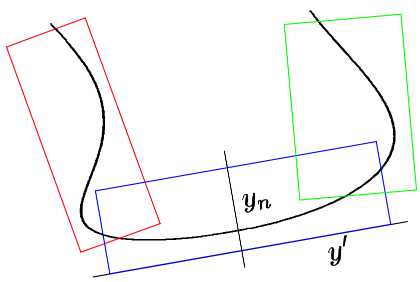
Clearly this cannot be a norm on $W^{k,p}(\Omega)$ for $k \geq 1$ since $|||1|||_{W_0^{k,p}(\Omega)} = 0$. Nevertheless, a result of Poincaré saves our day. Let us first fix the meaning of C^1-boundary or manifold.

Definition 2.1.8 *We will call a bounded $(n-1)$-dimensional manifold $\mathcal{M}$ a C^1-manifold if there are finitely many C^1-maps $f_i : \bar{A}_i \to \mathbb{R}$ with A_i an open set in $\mathbb{R}^{n-1}$ and corresponding Cartesian coordinates $\left\{y_1^{(i)}, \ldots, y_n^{(i)}\right\}$, say $i \in \{1, m\}$, such that*

$$\mathcal{M} = \bigcup_{i=1}^{m} \left\{ y_n^{(i)} = f_i(y_1^{(i)}, \ldots, y_{n-1}^{(i)}); (y_1^{(i)}, \ldots, y_{n-1}^{(i)}) \in \bar{A}_i \right\}.$$

We may assume there are open blocks $B_i := A_i \times (a_i, b_i)$ that cover $\mathcal{M}$ and are such that

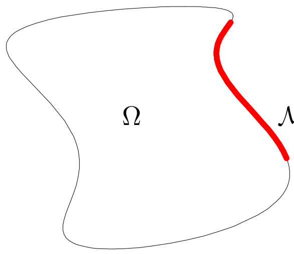
$$B_i \cap \mathcal{M} = \left\{ y_n^{(i)} = f_i(y_1^{(i)}, \ldots, y_{n-1}^{(i)}); (y_1^{(i)}, \ldots, y_{n-1}^{(i)}) \in \bar{A}_i \right\}.$$



Three blocks with local cartesian coordinates

Theorem 2.1.9 (A Poincaré type inequality) *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\ell \in \mathbb{R}^+$. Suppose that there is a bounded $(n-1)$-dimensional C^1-manifold $\mathcal{M} \subset \bar{\Omega}$ such that for every point $x \in \Omega$ there is a point $x^* \in \mathcal{M}$ connected by a smooth curve within Ω with length and curvature uniformly bounded, say by ℓ. Then there is a constant c such that for all $u \in C^1(\bar{\Omega})$ with $u = 0$ on $\mathcal{M}$*

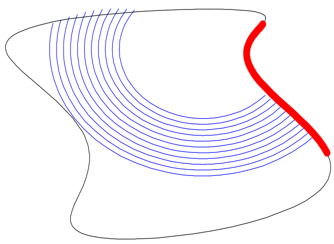
$$\int_\Omega |u|^p \, dx \le c \int_\Omega |\nabla u|^p \, dx.$$



Proof. First let us consider the one-dimensional problem. Taking $u \in C^1(0,\ell)$ with $u(0) = 0$ we find with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\begin{aligned}
\int_0^\ell |u(x)|^p \, dx &= \int_0^\ell \left| \int_0^x u'(s) ds \right|^p dx \\
&\le \int_0^\ell \left(\int_0^x |u'(s)|^p \, ds \right) \left(\int_0^x 1 ds \right)^{\frac{p}{q}} dx \\
&\le \left(\int_0^\ell |u'(s)|^p \, ds \right) \int_0^\ell x^{\frac{p}{q}} dx \;=\; \frac{1}{p} \ell^p \int_0^\ell |u'(s)|^p \, ds.
\end{aligned}$$

For the higher dimensional problem we go over to new coordinates. Let $\tilde{y}$ denote a system of coordinates on (part) of the manifold $\mathcal{M}$ and let y_n fill up the remaining direction. We may choose several systems of coordinates to fill up Ω. Since $\mathcal{M}$ is C^1 and $(n-1)$-dimensional we may assume that the Jacobian of each of these transformations is bounded from above and away from zero, let us assume $0 < c_1^{-1} < J < c_1$.



By these curvilinear coordinates we find for any number $\kappa > \ell$

$$\begin{aligned}
\int_{\text{shaded area}} |u(x)|^p \, dx &\leq c_1 \int_{\text{part of } \mathcal{M}} \left(\int_0^\kappa |u(y)|^p \, dy_n \right) d\tilde{y} \leq \\
&\leq c_1 \int_{\text{part of } \mathcal{M}} \frac{1}{p} \kappa^p \left(\int_0^\kappa \left| \frac{\partial}{\partial y_n} u(y) \right|^p dy_n \right) d\tilde{y} \leq \\
&\leq c_1 \int_{\text{part of } \mathcal{M}} \frac{1}{p} \kappa^p \left(\int_0^\kappa |(\nabla u)(y)|^p \, dy_n \right) d\tilde{y} \leq \\
&\leq c_1^2 \frac{1}{p} \kappa^p \int_{\text{shaded area}} |(\nabla u)(x)|^p \, dx.
\end{aligned}$$

Filling up Ω appropriately yields the result. ■

Exercise 18 *Show that the result of Theorem 2.1.9 indeed implies that there are constants $c_1, c_2 \in \mathbb{R}^+$ such that*

$$c_1 \, |||u|||_{W_0^{1,2}(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} \leq c_2 \, |||u|||_{W_0^{1,2}(\Omega)} .$$

Exercise 19 *Suppose that $u \in C^2([0,1]^2)$ with $u = 0$ on $\partial((0,1)^2)$. Show the following:*
there exists $c_1 > 0$ such that

$$\int_{[0,1]^2} u^2 \; dx \leq c_1 \int_{[0,1]^2} \left(u_{xx}^2 + u_{yy}^2 \right) \; dx.$$

Exercise 20 *Suppose that $u \in C^2([0,1]^2)$ with $u = 0$ on $\partial((0,1)^2)$. Show the following:*
there exists $c_2 > 0$ such that

$$\int_{[0,1]^2} u^2 \; dx \leq c_2 \int_{[0,1]^2} (\Delta u)^2 \, dx.$$

Hint: show that for these functions u the following holds:

$$\int_{[0,1]^2} u_{xx} u_{yy} \; dx = \int_{[0,1]^2} (u_{xy})^2 \; dx \geq 0.$$

Exercise 21 *Suppose that $u \in C^2([0,1]^2)$ with $u(0,x_2) = u(1,x_2) = 0$ for $x_2 \in [0,1]$. Show the following:*
there exists $c_3 > 0$ such that

$$\int_{[0,1]^2} u^2 \; dxdy \leq c_3 \int_{[0,1]^2} \left(u_{xx}^2 + u_{yy}^2 \right) \; dx.$$

Exercise 22 *Suppose that $u \in C^2([0,1]^2)$ with $u(0,x_2) = u(1,x_2) = 0$ for $x_2 \in [0,1]$. Prove or give a counterexample to:*
there exists $c_4 > 0$ such that

$$\int_{[0,1]^2} u^2 \; dxdy \leq c_4 \int_{[0,1]^2} (\Delta u)^2 \; dx.$$

Exercise 23 *Here are three spaces:* $W^{2,2}(0,1)$, $W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$ *and* $W_0^{2,2}(0,1)$, *which are all equipped with the norm* $\|\cdot\|_{W^{2,2}(0,1)}$. *Consider the functions* $f_\alpha(x) = x^\alpha (1-x)^\alpha$ *for* $\alpha \in \mathbb{R}$. *Complete the next sentences:*

1. *If* α *....., then* $f_\alpha \in W^{2,2}(0,1)$.
2. *If* α *....., then* $f_\alpha \in W^{2,2}(0,1) \cap W_0^{1,2}(0,1)$.
3. *If* α *....., then* $f_\alpha \in W_0^{2,2}(0,1)$.

2.2 Restricting and extending

If $\Omega \subset \mathbb{R}^n$ is bounded and $\partial\Omega \in C^1$ then by assumption we may split the boundary in finitely many pieces, $i = 1, \dots, m$, and we may describe each of these by

$$\left\{ y_n^{(i)} = f_i(y_1^{(i)}, \dots, y_{n-1}^{(i)}); (y_1^{(i)}, \dots, y_{n-1}^{(i)}) \in A_i \right\}$$

for some bounded open set $A_i \in \mathbb{R}^{n-1}$. We may even impose higher regularity of the boundary:

Definition 2.2.1 *For a bounded domain Ω we say $\partial\Omega \in C^{m,\alpha}$ if there is a finite set of functions f_i, open sets A_i and Cartesian coordinate systems as in Definition 2.1.8, and if moreover $f_i \in C^{m,\alpha}(\bar{A}_i)$.*

First let us consider multiplication operators.

Lemma 2.2.2 *Fix $\Omega \subset \mathbb{R}^n$ bounded and $\zeta \in C_0^\infty(\mathbb{R}^n)$. Consider the multiplication operator $M(u) = \zeta u$. This operator M is bounded in any of the spaces $C^m(\bar{\Omega})$, $C^{m,\alpha}(\bar{\Omega})$ with $m \in \mathbb{N}$ and $\alpha \in (0, 1]$, $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ with $m \in \mathbb{N}$ and $p \in (1, \infty)$.*

Exercise 24 *Prove this lemma.*

Often it is convenient to study the behaviour of some function locally. One of the tools is the so-called partition of unity.

Definition 2.2.3 *A set of functions $\{\zeta_i\}_{i=1}^{\ell}$ is called a C^∞-partition of unity for Ω if the following holds:*

1. *$\zeta_i \in C_0^\infty(\mathbb{R}^n)$ for all i;*
2. *$0 \leq \zeta_i(x) \leq 1$ for all i and $x \in \Omega$;*
3. *$\sum_{i=1}^{\ell} \zeta_i(x) = 1$ for $x \in \Omega$.*

Sometimes it is useful to work with a partition of unity of Ω that coincides with the coordinate systems mentioned in Definition 2.1.8 and 2.2.1. In fact it is possible to find a partition of unity $\{\zeta_i\}_{i=1}^{\ell+1}$ such that

$$\begin{cases} \text{support}(\zeta_i) \subset B_i \text{ for } i = 1, \dots, \ell, \\ \text{support}(\zeta_{\ell+1}) \subset \Omega. \end{cases} \tag{2.1}$$

Any function u defined on Ω can now be written as

$$u = \sum_{i=1}^{\ell} \zeta_i u$$

and by Lemma 2.2.2 the functions $u_i := \zeta_i u$ have the same regularity as u but are now restricted to an area where we can deal with them. Indeed we find $\text{support}(u_i) \subset \text{support}(\zeta_i)$.

Definition 2.2.4 *A mollifier on $\mathbb{R}^n$ is a function J_1 with the following properties:*

1. $J_1 \in C^\infty(\mathbb{R}^n)$;

2. $J_1(x) \geq 0$ *for all* $x \in \mathbb{R}^n$;

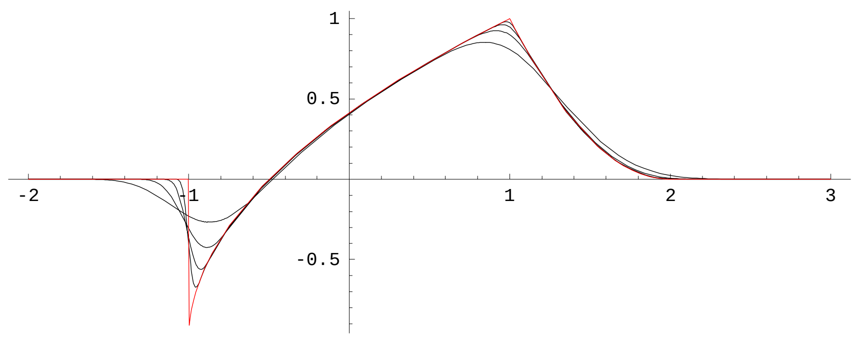
3. *support*$(J_1) \subset \{x \in \mathbb{R}^n; |x| \leq 1\}$;

4. $\int_{\mathbb{R}^n} J_1(x)dx = 1$.

By rescaling one arrives at the family of functions $J_\varepsilon(x) = \varepsilon^{-n} J_1(\varepsilon^{-1}x)$ that goes in some sense to the δ-function in 0. The nice property is the following. If u has compact support in Ω we may extend u by 0 outside of Ω. Then, for $u \in C^{m,\alpha}(\bar{\Omega})$ respectively $u \in W^{m,p}(\Omega)$ we may define

$$u_\varepsilon(x) := (J_\varepsilon * u)(x) = \int_{y \in \mathbb{R}^n} J_\varepsilon(x-y)u(y)dy,$$

to find a family of C^∞-functions that approximates u when $\varepsilon \downarrow 0$ in $\|\cdot\|_{C^{m,\alpha}(\Omega)}$ respectively in $\|\cdot\|_{W^{m,p}(\Omega)}$-norm.



Mollyfying a jump in u, u' and u''.

Example 2.2.5 *The standard example of a mollifier is the function*

$$\rho(x) = \begin{cases} c \exp\left(\frac{-1}{1-|x|^2}\right) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where the c is fixed by condition 4 in Definition 2.2.4.

There are other ways of restricting but let us now focus on extending a function. Without any assumption on the boundary this can become very involved. For bounded Ω with at least $\partial\Omega \in C^1$ we may define a bounded extension operator. But first let us consider extension operators from $C^m[0,1]$ to $C^m[-1,1]$.

For $m = 0$ one proceeds by a simple reflection. Set

$$E_0(u)(x) = \begin{cases} u(x) & \text{for } \quad 0 \leq x \leq 1, \\ u(-x) & \text{for } -1 \leq x < 0, \end{cases}$$

and one directly checks that $E_0(u) \in C[-1,1]$ for $u \in C[0,1]$ and moreover

$$\|E_0(u)\|_{C[-1,1]} \leq \|u\|_{C[0,1]}.$$

A simple and elegant procedure does it for $m > 0$. Set

$$E_m(u)(x) = \begin{cases} u(x) & \text{for } \quad 0 \le x \le 1, \\ \sum_{k=1}^{m+1} \alpha_{m,k} u(-\frac{1}{k}x) & \text{for } -1 \le x < 0, \end{cases}$$

and compute the coefficients $\alpha_{m,k}$ from the equations coming out of

$$(E_m(u))^{(j)} \left(0^+\right) = (E_m(u))^{(j)} \left(0^-\right),$$

namely

$$1 = \sum_{k=1}^{m+1} \alpha_{m,k} \left(-\frac{1}{k}\right)^j \text{ for } j = 0, 1, \dots, m. \tag{2.2}$$

Then $E_m(u) \in C^m[-1,1]$ for $u \in C^m[0,1]$ and moreover with $c_m = \sum_{k=1}^{m+1} |\alpha_{m,k}|$:

$$\|E_m(u)\|_{C^m[-1,1]} \le c_m \|u\|_{C^m[0,1]}.$$

Exercise 25 *Show that there is $c_{p,m} \in \mathbb{R}^+$ such that for all $u \in C^m[0,1]$:*

$$\|E_m(u)\|_{W^{m,p}(-1,1)} \le c_{p,m} \|u\|_{W^{m,p}(0,1)}.$$

Exercise 26 *Let $\Omega \subset \mathbb{R}^{n-1}$ be a bounded domain. Convince yourself that*

$$E_m(u)(x', x_n) = \begin{cases} u(x', x_n) & \textit{for } \quad 0 \le x_n \le 1, \\ \sum_{k=1}^{m+1} \alpha_{m,k} u(x', -\frac{1}{k}x_n) & \textit{for } -1 \le x_n < 0, \end{cases}$$

is a bounded extension operator from $C^m(\Omega \times [0,1])$ to $C^m(\Omega \times [-1,1])$.

Lemma 2.2.6 *Let Ω and Ω' be bounded domains in $\mathbb{R}^n$ and $\partial\Omega \in C^m$ with $m \in \mathbb{N}^+$. Suppose that $\bar{\Omega} \subset \Omega'$. Then there exists a linear extension operator E_m from $C^m(\bar{\Omega})$ to $C_0^m(\bar{\Omega}')$ such that:*

1. $(E_m u)_{|\Omega} = u$,
2. $(E_m u)_{|\mathbb{R}^n \setminus \Omega'} = 0$
3. *there is $c_{\Omega,\Omega',m}$ and $c_{\Omega,\Omega',m,p} \in \mathbb{R}^+$ such that for all $u \in C^m(\bar{\Omega})$*

$$\|E_m u\|_{C^m(\bar{\Omega}')} \le c_{\Omega,\Omega',m} \|u\|_{C^m(\bar{\Omega})}, \tag{2.3}$$

$$\|E_m u\|_{W^{m,p}(\Omega')} \le c_{\Omega,\Omega',m,p} \|u\|_{W^{m,p}(\Omega)}. \tag{2.4}$$

Proof. Let the A_i, f_i, B_i and ζ_i be as in Definition 2.1.8 and (2.1). Let us consider $u_i = \zeta_i u$ in the coordinates $\{y_1^{(i)}, \dots, y_n^{(i)}\}$. First we will flatten this part of the boundary by replacing the last coordinate by $y_n^{(i)} = \tilde{y}_n^{(i)} + f_i(y_1^{(i)}, \dots, y_{n-1}^{(i)})$. Writing u_i in these new coordinates $\{y_*^{(i)}, \tilde{y}_n^{(i)}\}$, with $y_*^{(i)} = (y_1^{(i)}, \dots, y_{n-1}^{(i)})$ and setting B_i^* the transformed box, we will first assume that u_i is extended by zero for $\tilde{y}_n^{(i)} > 0$ and large. Now we may define the extension of u_i by

$$\bar{u}_i(y_*^{(i)}, \tilde{y}_n^{(i)}) = \begin{cases} u_i(y_*^{(i)}, \tilde{y}_n^{(i)}) & \text{if } \tilde{y}_n^{(i)} \ge 0, \\ \sum_{k=1}^{m+1} \alpha_k u_i(y_*^{(i)}, -\frac{1}{k}\tilde{y}_n^{(i)}) & \text{if } \tilde{y}_n^{(i)} < 0, \end{cases}$$

where the α_k are defined by the equations in (2.2). Assuming that $u \in C^m(\bar{\Omega})$ we find that with the coefficients chosen this way the function $\bar{u}_i \in C^m(\bar{B}_i^*)$ and its derivatives of order $\leq m$ are continuous. Moreover, also after the reverse transformation the function $\bar{u}_i \in C^m(\bar{B}_i)$. Indeed this fact remains due to the assumptions on f_i. It is even true that there is a constant $C_{i,\partial\Omega}$, which just depend on the properties of the i^{th} local coordinate system and ζ_i, such that

$$\|\bar{u}_i\|_{W^{m,p}(B_i)} \leq C_{i,\partial\Omega} \|u_i\|_{W^{m,p}(B_i\cap\Omega)}. \tag{2.5}$$

An approximation argument show that (2.5) even holds for all $u \in W^{m,p}(\Omega)$. Up to now we have constructed an extension operator $\tilde{E}_m : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ by

$$\tilde{E}_m(u) = \sum_{i=1}^{\ell} \bar{u}_i + \zeta_{\ell+1} u. \tag{2.6}$$

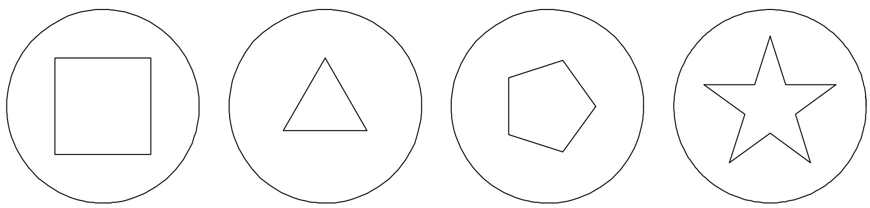
As a last step let χ be a C_0^∞-function with such that $\chi(x) = 1$ for $x \in \bar{\Omega}$ and support$(\chi) \subset \Omega'$. We define the extension operator E_m by

$$E_m(u) = \chi \sum_{i=1}^{\ell} \bar{u}_i + \zeta_{\ell+1} u.$$

Notice that we repeatedly used Lemma 2.2.2 in order to obtain the estimate in (2.4).

It remains to show that $E_m(u) \in W_0^{m,p}(\Omega')$. Due to the assumption that support$(\chi) \subset \Omega'$ we find that support$(J_\varepsilon * E_m(u)) \subset \Omega'$ for ε small enough where J_ε is a mollifier. Hence $(J_\varepsilon * E_m(u)) \in C_0^\infty(\Omega')$ and since $J_\varepsilon * E_m(u)$ approaches $E_m u$ in $W^{m,p}(\Omega')$-norm for $\varepsilon \downarrow 0$ one finds $E_m u \in W_0^{m,p}(\Omega')$. ■

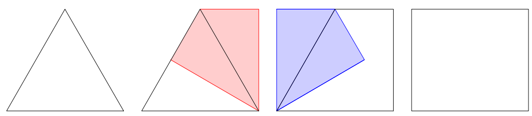
Exercise 27 *A square inside a disk:* $\Omega_1 = (-1,1)^2$ *and* $\Omega_1' = \left\{x \in \mathbb{R}^2; |x| < 2\right\}$. *Construct a bounded extension operator* $E : W^{2,2}(\Omega_1) \to W_0^{2,2}(\Omega_1')$.



Exercise 28 *Think about the similar question in case of*

an equilateral triangle:	$\Omega_2 = \left\{(x,y)\,; -\frac{1}{2} < y < 1 - \sqrt{3}\,|x|\right\}$,
a regular pentagon:	$\Omega_3 = co\left\{(\cos(2k\pi/5), \sin(2k\pi/5)); k = 0,1,\ldots,4\right\}^o$.
a US-army star	$\Omega_4 = ...$

$co(A)$ is the convex hull of A; the small ^{o} means the open interior.



Hint for the triangle.

Above we have defined a mollifier on $\mathbb{R}^n$. If we want to use such a mollifier on a bounded domain a difficulty arises when u doesn't have a compact support in Ω. In that case we have to use the last lemma first and extend the function. Notice that we don't need the last part of the proof of that lemma but may use $\tilde{E}_m$ defined in (2.6).

Exercise 29 *Let Ω and Ω' be bounded domains in $\mathbb{R}^n$ with $\bar{\Omega} \subset \Omega'$ and let $\varepsilon > 0$.*

1. *Suppose that $u \in C(\bar{\Omega}')$. Prove that $u_\varepsilon(x) := \int_{y \in \Omega'} J_\varepsilon(x-y)u(y)dy$ is such that $u_\varepsilon \to u$ in $C(\bar{\Omega})$ for $\varepsilon \downarrow 0$.*

2. *Let $\gamma \in (0,1]$ and suppose that $u \in C^\gamma(\bar{\Omega}')$. Prove that $u_\varepsilon \to u$ in $C^\gamma(\bar{\Omega})$ for $\varepsilon \downarrow 0$.*

3. *Let $u \in C^k(\bar{\Omega}')$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$. Show that $\left(\frac{\partial}{\partial x}\right)^\alpha u_\varepsilon = \left(\left(\frac{\partial}{\partial x}\right)^\alpha u\right)_\varepsilon$ in Ω if $\varepsilon < \inf\{|x-x'|\,; x \in \partial\Omega,\ x' \in \partial\Omega'\}$.*

Exercise 30 *Let Ω be bounded domains in $\mathbb{R}^n$ and let $\varepsilon > 0$.*

1. *Derive from Hölder's inequality that*

$$\int ab\ dx \leq \left(\int |a|\ dx\right)^{\frac{1}{q}} \left(\int |a|\,|b|^p\ dx\right)^{\frac{1}{p}} \text{ for } p,q \in (1,\infty) \text{ with } \frac{1}{p}+\frac{1}{q}=1.$$

2. *Suppose that $u \in L^p(\Omega)$ with $p < \infty$. Set $\bar{u}(x) = u(x)$ for $x \in \Omega$ and $\bar{u}(x) = 0$ elsewhere. Prove that $u_\varepsilon(x) := \int_{y \in \mathbb{R}^{n\prime}} J_\varepsilon(x-y)\bar{u}(y)dy$ satisfies*

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}\,.$$

3. *Use that $C(\bar{\Omega}')$ is dense in $L^p(\Omega')$ and the previous exercise to conclude that $u_\varepsilon \to u$ in $L^p(\Omega)$.*

2.3 Traces

We have seen that $u \in W_0^{1,2}(\Omega)$ means $u = 0$ on $\partial\Omega$ in some sense, but not $\frac{\partial}{\partial n}u$ being zero on the boundary although we defined $W_0^{1,2}(\Omega)$ by approximation through functions in $C_0^\infty(\Omega)$. What can we do if the boundary data we are interested in are nonhomogeneous? The trace operator solves that problem.

Theorem 2.3.1 *Let $p \in (1,\infty)$ and assume that Ω is bounded and $\partial\Omega \in C^1$.*

A. For $u \in W^{1,p}(\Omega)$ let $\{u_m\}_{m=1}^\infty \subset C^1(\bar{\Omega})$ be such that $\|u - u_m\|_{W^{1,p}(\Omega)} \to 0$. Then $u_{m|\partial\Omega}$ converges in $L^p(\partial\Omega)$, say to $v \in L^p(\partial\Omega)$, and the limit v only depends on u.

So $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ with $Tu = v$ is well defined.

B. The following holds:

1. *T is a bounded linear operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$;*
2. *if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, then $Tu = u_{|\partial\Omega}$.*

Remark 2.3.2 *This T is called the trace operator. Bounded means there is $C_{p,\Omega}$ such that*

$$\|Tu\|_{L^p(\partial\Omega)} \le C_{p,\Omega} \|u\|_{W^{1,p}(\Omega)}. \tag{2.7}$$

Proof. We may assume that there are finitely many maps and sets of Cartesian coordinate systems as in Defintion 2.1.8. Let $\{\zeta_i\}_{i=1}^m$ be a C^∞ partition of unity for Ω, that fits with the finitely many boundary maps:

$$\partial\Omega \cap \mathrm{support}(\zeta_i) \subset \partial\Omega \cap \left\{ y_n^{(i)} = f_i(y_1^{(i)}, \dots, y_{n-1}^{(i)}); (y_1^{(i)}, \dots, y_{n-1}^{(i)}) \in A_i \right\}.$$

We may choose such a partition since these local coordinate systems do overlap; the A_i are open. Now for the sake of simpler notation let us assume these boundary parts are even flat. For this assumption one will pay by a factor $(1 + |\nabla f_i|^2)^{1/2}$ in the integrals but by the C^1-assumption for $\partial\Omega$ the values of $|\nabla f_i|^2$ are uniformly bounded.

First let us suppose that $u \in C^1(\bar{\Omega})$ and derive an estimate as in (2.7). Integrating inwards from the (flat) boundary part $A_i \subset \mathbb{R}^{n-1}$ gives, with ℓ 'outside the support of ζ_i', that

$$\begin{aligned}
\int_{A_i} \zeta_i |u|^p \, dx' &= -\int_{A_i} \int_0^\ell \frac{\partial}{\partial x_n} (\zeta_i |u|^p) \, dx_n dx' \\
&= -\int_{A_i \times (0,\ell)} \left(\frac{\partial \zeta_i}{\partial x_n} |u|^p + p\zeta_i |u|^{p-2} u \frac{\partial u}{\partial x_n} \right) dx \\
&\le C \int_{A_i \times (0,\ell)} \left(|u|^p + |u|^{p-1} \left| \frac{\partial u}{\partial x_n} \right| \right) dx \\
&\le C \frac{2p-1}{p} \int_{A_i \times (0,\ell)} |u|^p \, dx + C \frac{1}{p} \int_{A_i \times (0,\ell)} \left| \frac{\partial u}{\partial x_n} \right|^p dx \\
&\le \tilde{C} \int_{A_i \times (0,\ell)} (|u|^p + |\nabla u|^p) \, dx.
\end{aligned}$$

In the one but last step we used Young's inequality:

$$ab \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \text{ for } p, q > 0 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \tag{2.8}$$

Combining such an estimate for all boundary parts we find, adding finitely many constants, that

$$\int_{\partial\Omega} |u|^p \, dx' \leq C \int_\Omega (|u|^p + |\nabla u|^p) \, dx.$$

So

$$\left\| u_{|\partial\Omega} \right\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \text{ for } u \in C^1\left(\bar{\Omega}\right), \tag{2.9}$$

and if $u_m \in C^1(\Omega)$ with $u_m \to u$ in $W^{1,p}(u)$ then $\left\| u_{|\partial\Omega} - u_{m|\partial\Omega} \right\|_{L^p(\partial\Omega)} \leq C \|u - u_m\|_{W^{1,p}(\Omega)} \to 0$. So for $u \in C^1(\bar{\Omega})$ the operator T is well-defined and $Tu = u_{|\partial\Omega}$.

Now assume that $u \in W^{1,p}(\Omega)$ without the $C^1\left(\bar{\Omega}\right)$ restriction. Then there are $C^\infty(\bar{\Omega})$ functions u_m converging to u and

$$\|Tu_m - Tu_k\|_{L^p(\partial\Omega)} \leq C \|u_m - u_k\|_{W^{1,p}(\Omega)}. \tag{2.10}$$

So $\{Tu_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(\partial\Omega)$ and hence converges. By (2.9) one also finds that the limit does not depend on the chosen sequence: if $u_m \to u$ and $v_m \to u$ in $W^{1,p}(\Omega)$, then $\|Tu_m - Tv_m\|_{L^p(\partial\Omega)} \leq C \|u_m - v_m\|_{W^{1,p}(\Omega)} \to 0$. So $Tu := \lim_{m\to\infty} Tu_m$ is well defined in $L^p(\partial\Omega)$ for all $u \in W^{1,p}(\Omega)$.

The fact that T is bounded also follows from (2.9):

$$\|Tu\|_{L^p(\partial\Omega)} = \lim_{m\to\infty} \|Tu_m\|_{L^p(\partial\Omega)} \leq C \lim_{m\to\infty} \|u_m\|_{W^{1,p}(\Omega)} = C \|u\|_{W^{1,p}(\Omega)}.$$

It remains to show that if $u \in W^{1,p}(\Omega) \cap C\left(\bar{\Omega}\right)$ then $Tu = u_{|\partial\Omega}$. To complete that argument we use that functions in $W^{1,p}(\Omega) \cap C\left(\bar{\Omega}\right)$ can be approximated by a sequence of $C^1(\bar{\Omega})$-functions $\{u_m\}_{m=1}^\infty$ in $W^{1,p}(\Omega)$-sense and such that $u_m \to u$ uniformly in $\bar{\Omega}$. For example, since $\partial\Omega \in C^1$ one may extend functions in a small ε_1-neighborhood outside of $\bar{\Omega}$ by a local reflection as in Lemma 2.2.6. Setting $\tilde{u} = E(u)$ and one finds $\tilde{u} \in W^{1,p}(\Omega_\varepsilon) \cap C\left(\overline{\Omega_\varepsilon}\right)$ and $\|\tilde{u}\|_{W^{1,p}(\Omega_\varepsilon)} \leq C \|u\|_{W^{1,p}(\Omega)}$. A mollifier allows us to construct a sequence $\{u_m\}_{m=1}^\infty$ in $C^\infty(\bar{\Omega})$ that approximates both u in $W^{1,p}(\Omega)$ as well as uniformly on $\bar{\Omega}$.

So

$$\begin{aligned} \left\| u_{|\partial\Omega} - u_{m|\partial\Omega} \right\|_{L^p(\partial\Omega)} &\leq |\partial\Omega|^{1/p} \left\| u_{|\partial\Omega} - u_{m|\partial\Omega} \right\|_{C(\partial\Omega)} \leq \\ &\leq |\partial\Omega|^{1/p} \|u - u_m\|_{C(\bar{\Omega})} \to 0 \end{aligned}$$

and $\left\| Tu - u_{m|\partial\Omega} \right\|_{L^p(\partial\Omega)} \to 0$ implying that $Tu = u_{|\partial\Omega}$. ∎

Exercise 31 *Prove Young's inequality (2.8).*

2.4 Zero trace and $W_0^{1,p}(\Omega)$

So by now there are two ways of considering boundary conditions. First there is $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $\|\cdot\|_{W^{1,p}(\Omega)}$-norm and secondly the trace operator. It would be rather nice if for zero boundary conditions these would coincide.

Theorem 2.4.1 *Fix $p \in (1,\infty)$ and suppose that Ω is bounded with $\partial\Omega \in C^1$. Suppose that T is the trace operator from Theorem 2.3.1. Let $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \text{ if and only if } Tu = 0 \text{ on } \partial\Omega.$$

Proof. ($\Rightarrow$) If $u \in W_0^{1,p}(\Omega)$ then we may approximate u by $u_m \in C_0^\infty(\Omega)$ and since T is a bounded linear operator:

$$\|Tu\|_{L^p(\Omega)} = \|Tu - Tu_m\|_{L^p(\Omega)} \le c\,\|u-u_m\|_{W^{1,p}(\Omega)} \to 0.$$

($\Leftarrow$) Assuming that $Tu = 0$ on $\partial\Omega$ we have to construct a sequence in $C_0^\infty(\Omega)$ that converges to u in the $\|\cdot\|_{W^{1,p}(\Omega)}$-norm. Since $\partial\Omega \in C^1$ we again may work on the finitely many local coordinate systems as in Definition 2.2.1. We start by fixing a partition of unity $\{\zeta_i\}_{i=1}^m$ as in the proof of the previous theorem and which again corresponds with the local coordinate systems. Next we assume the boundary sets are flat, say $\partial\Omega \cap B_i = \Gamma_{B_i} \subset \mathbb{R}^{n-1}$ and set $\Omega \cap B_i = \Omega_{B_i}$ where B_i is a box.

Since $Tu = 0$ we may take any sequence $\{u_k\}_{k=1}^\infty \in C^1(\bar{\Omega})$ such that $\|u_k - u\|_{W^{1,p}(\Omega)} \to 0$ and it follows from the definition of T that $u_{m|\partial\Omega} = Tu_m \to 0$. Now we proceed in three steps. From $v(t) = v(0) + \int_0^t v'(s)ds$ one gets

$$|v(t)| \le |v(0)| + \int_0^t |v'(s)|\,ds$$

and hence by Minkowski's and Young's inequality

$$|v(t)|^p \le c_p\,|v(0)|^p + c_p\left(\int_0^t |v'(s)|\,ds\right)^p \le c_p\,|v(0)|^p + c_p t^{p-1}\int_0^t |v'(s)|^p\,ds.$$

It implies that

$$\begin{aligned} &\int_{\Gamma_B} |u_m(x',x_n)|^p\,dx' \le \\ \le\ & c_p\int_{\Gamma_B} |u_m(x',0)|^p\,dx' + c_p x_n^{p-1}\int_{\Gamma_B\times[0,x_n]}\left|\frac{\partial}{\partial x_n}u_m(x',s)\right|^p dx'ds, \end{aligned}$$

and since $u_{m|\partial\Omega} \to 0$ in $L^p(\partial\Omega)$ and $u_m \to u$ in $W^{1,p}(\Omega)$ we find that

$$\int_{\Gamma_B} |u(x',x_n)|^p\,dx' \le c_p x_n^{p-1}\int_{\Gamma_B\times[0,x_n]}\left|\frac{\partial}{\partial x_n}u(x',s)\right|^p dx'ds. \tag{2.11}$$

Taking $\zeta \in C_0^\infty(\mathbb{R})$ such that

$$\begin{array}{ll} \zeta(x) = 1 & \text{for } x \in [0,1], \\ \zeta(x) = 0 & \text{for } x \notin [-1,2], \\ 0 \le \zeta \le 1 & \text{in } \mathbb{R}, \end{array}$$

we set

$$w_m(x', x_n) = (1 - \zeta(mx_n))\, u(x', x_n).$$

One directly finds for $m \to \infty$ that

$$\int_{\Omega_B} |u - w_m|^p \, dx \le \int_{\Gamma_B \times [0, 2m^{-1}]} |u(x)|^p \, dx \to 0.$$

For the gradient part of the $\|\cdot\|$-norm we find

$$\begin{aligned} & \int_{\Omega_B} |\nabla u - \nabla w_m|^p \, dx = \int_{\Omega_B} |\nabla(\zeta(mx_n)\, u(x))|^p \, dx \\ \le \quad & 2^{p-1} \int_{\Omega_B} \left(\zeta(mx_n)^p \, |\nabla u(x)|^p + m^p \left|\zeta'(mx_n)\right|^p |u(x)|^p \right) dx. \quad (2.12) \end{aligned}$$

For the first term of (2.12) when $m \to \infty$ we obtain

$$\int_{\Omega_B} \zeta(mx_n)^p \, |\nabla u(x)|^p \, dx \le \int_{\Gamma_B \times [0, 2m^{-1}]} |\nabla u(x)|^p \, dx \to 0$$

and from (2.11) for the second term of (2.12) when $m \to \infty$

$$\begin{aligned} & \int_{\Omega_B} m^p \left|\zeta'(mx_n)\right|^p |u(x)|^p \, dx = \int_0^{2m^{-1}} \int_{\Gamma_B} m^p \left|\zeta'(mx_n)\right|^p |u(x)|^p \, dx' dx_n \\ \le \quad & C_{p,\zeta} m^p \int_0^{2m^{-1}} x_n^{p-1} \left(\int_{\Gamma_B \times [0, 2m^{-1}]} \left| \frac{\partial}{\partial x_n} u(x', t) \right|^p dx' dt \right) dx_n \\ \le \quad & C'_{p,\zeta} \int_{\Gamma_B \times [0, 2m^{-1}]} \left| \frac{\partial}{\partial x_n} u(x', t) \right|^p dx' dt \to 0. \end{aligned}$$

So $w_m \to u$ in $\|\cdot\|_{W^{1,p}(\Omega)}$-norm and moreover support(w_n) lies compactly in 'Ω'. (Here we have to go back to the curved boundaries and glue the $w_m^{(i)}$ together). Now there is room for using a mollifier. With J_1 as before we may find that for all $\varepsilon < \frac{1}{2m}$ (in Ω this becomes $\varepsilon < \frac{1}{cm}$ for some uniform constant depending on $\partial\Omega$) that $v_{\varepsilon,m} = J_\varepsilon * w_m$ is a $C_0^\infty(\Omega)$-function. We may choose a sequence of ε_m such that $\|v_{\varepsilon_m,m} - w_m\|_{W^{1,p}(\Omega)} \le \frac{1}{m}$ and hence $v_{\varepsilon,m} \to u$ in $W^{1,p}(\Omega)$. ■

Exercise 32 *Prove Minkowski's inequality: for all $\varepsilon > 0$, $p \in (1, \infty)$ and $a, b \in \mathbb{R}$:*

$$|a + b|^p \le (1 + \varepsilon)^{p-1} |a|^p + \left(1 + \frac{1}{\varepsilon}\right)^{p-1} |b|^p.$$

2.5 Gagliardo, Nirenberg, Sobolev and Morrey

In the previous sections we have (re)visited some different function spaces. For bounded (smooth) domains there are the more obvious relations that $W^{m,p}(\Omega) \subset W^{k,p}(\Omega)$ when $m > k$ and $W^{m,p}(\Omega) \subset W^{m,q}(\Omega)$ when $p > q$, not to mention $C^{m,\alpha}(\bar{\Omega}) \subset C^{m,\beta}(\bar{\Omega})$ when $\alpha > \beta$ or $C^m(\bar{\Omega}) \subset W^{m,p}(\Omega)$. But what about the less obvious relations? Are there conditions such that a Sobolev space is contained in some Hölder space? Or is it possible that $W^{m,p}(\Omega) \subset W^{k,q}(\Omega)$ with $m > k$ but $p < q$? Here we will only recall two of the most famous estimates that imply an answer to some of these questions.

Theorem 2.5.1 (Gagliardo-Nirenberg-Sobolev inequality) *Fix $n \in \mathbb{N}^+$ and suppose that $1 \leq p < n$. Then there exists a constant $C = C_{p,n}$ such that*

$$\|u\|_{L^{\frac{pn}{n-p}}(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

Theorem 2.5.2 (Morrey's inequality) *Fix $n \in \mathbb{N}^+$ and suppose that $n < p < \infty$. Then there exists a constant $C = C_{p,n}$ such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$.

Remark 2.5.3 *These two estimates form a basis for most of the Sobolev imbedding Theorems. Let us try to supply you with a way to remember the correct coefficients. Obviously for $W^{k,p}(\Omega) \subset C^{\ell,\alpha}(\bar{\Omega})$ and $W^{k,p}(\Omega) \subset W^{\ell,q}(\Omega)$ (with $q > p$) one needs $k > \ell$. Let us introduce some number κ for the quality of the continuity. For the Hölder spaces $C^{m,\alpha}(\bar{\Omega})$ this is 'just' the coefficient:*

$$\kappa_{C^{m,\alpha}} = m + \alpha.$$

For the Sobolev spaces $W^{m,p}(\Omega)$ with $\Omega \in \mathbb{R}^n$, which functions have 'less differentiability' than the ones in $C^m\left(\bar{\Omega}\right)$, this less corresponds with $-\frac{1}{p}$ for each dimension and we set

$$\kappa_{W^{m,p}} = m - \frac{n}{p}.$$

For $X_1 \subset X_2$ it should hold that $\kappa_{X_1} \geq \kappa_{X_2}$. The optimal estimates appear in the theorems above; so comparing $W^{1,p}$ and L^q, respectively $W^{1,p}$ and C^α, one finds

$$\begin{cases} 1 - \frac{n}{p} \geq -\frac{n}{q} & \text{if } q \leq \frac{pn}{n-p} \text{ and } n > p, \\ 1 - \frac{n}{p} \geq \alpha & \text{if } n < p. \end{cases}$$

This is just a way of remembering the constants. In case of equality of the κ's an imbedding might hold or just fail.

Exercise 33 *Show that u defined on $B_1 = \{x \in \mathbb{R}^n; |x| < 1\}$ with $n > 1$ by $u(x) = \log(\log(1 + \frac{1}{|x|}))$ belongs to $W^{1,n}(B_1)$. It does not belong to $L^\infty(B_1)$.*

Exercise 34 *Set $\Omega = \{x \in \mathbb{R}^n; |x| < 1\}$ and define $u_\beta(x) = |x|^\beta$ for $x \in \Omega$. Compute for which β it holds that:*

1. $u_\beta \in C^{0,\alpha}(\bar{\Omega})$;

2. $u_\beta \in W^{1,p}(\Omega)$;

3. $u_\beta \in L^q(\Omega)$.

Exercise 35 *Set* $\Omega = \{x \in \mathbb{R}^n; |x - (1,0,\dots,0)| < 1\}$ *and define* $u_\beta(x) = |x|^\beta$ *for* $x \in \Omega$. *Compute for which* β *it holds that:*

1. $u_\beta \in C^{0,\alpha}(\partial\Omega)$;

2. $u_\beta \in W^{1,p}(\partial\Omega)$;

3. $u_\beta \in L^q(\partial\Omega)$.

Proof of Theorem 2.5.1. Since u has a compact support we have (using $f(t) = f(a) + \int_a^t f'(s)ds$)

$$|u(x)| = \left| \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots x_{i-1}, y_i, x_{i+1}, \dots x_n) dy_i \right| \leq \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i,$$

so that

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i \right)^{\frac{1}{n-1}}. \tag{2.13}$$

When integrating with respect to x_1 one finds that on the right side there are $n-1$ integrals that are x_1-dependent and one integral that does not depend on x_1. So we can get out one factor for free and use a generalized Hölder inequality for the $n-1$ remaining ones:

$$\int |c_1|^{\frac{1}{n-1}} |a_2|^{\frac{1}{n-1}} \dots |a_n|^{\frac{1}{n-1}} dt \leq \left(|c_1| \int |a_2| \, dt \dots \int |a_2| \, dt \right)^{\frac{1}{n-1}}.$$

For (2.13) it gives us

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 &\leq \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1} \right| dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i \right)^{\frac{1}{n-1}} dx_1 \\
&\leq \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1} \right| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i dx_1 \right)^{\frac{1}{n-1}} \\
&= \left(\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1} \right| dy_1 \prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i dx_1 \right)^{\frac{1}{n-1}}.
\end{aligned}$$

Repeating this argument one finds

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_1} \right| dy_1 dx_2 \right)^{\frac{1}{n-1}} \times \\
&\times \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_2} \right| dy_2 dx_1 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_i} \right| dy_i dx_1 dx_2 \right)^{\frac{1}{n-1}}
\end{aligned}$$

and after n steps

$$\|u\|_{L^{\frac{n}{n-1}}}^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1}^{\frac{1}{n-1}} \leq \|\nabla u\|_{L^1}^{\frac{n}{n-1}} \tag{2.14}$$

which completes the proof of Theorem 2.5.1 for $p = 1$. For $p \in (1, n)$ one uses (2.14) with u replaced by $|u|^\alpha$ for appropriate $\alpha > 1$. Indeed, again with Hölder and $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}
\|u\|_{L^{\alpha\frac{n}{n-1}}}^{\alpha} &= \||u|^\alpha\|_{L^{\frac{n}{n-1}}} \leq \|\nabla |u|^\alpha\|_{L^1} = \alpha \left\| |u|^{\alpha-1} |\nabla u| \right\|_{L^1} \\
&\leq \alpha \left\| |u|^{\alpha-1} \right\|_{L^q} \|\nabla u\|_{L^p} = \alpha \|u\|_{L^{(\alpha-1)q}}^{\alpha-1} \|\nabla u\|_{L^p}.
\end{aligned}$$

If we take $\alpha > 1$ such that $\alpha\frac{n}{n-1} = (\alpha - 1)q$, in other words $\alpha = \frac{q}{q-\frac{n}{n-1}}$ and $q > \frac{n}{n-1}$ should hold, we find $\|u\|_{L^{\alpha\frac{n}{n-1}}} \leq |\alpha| \|\nabla u\|_{L^p}$. Since $q > \frac{n}{n-1}$ coincides with $p < n$ and $\alpha\frac{n}{n-1} = \frac{np}{n-p}$ the claim follows. ■

Proof of Theorem 2.5.2. Let us fix $0 < s \leq r$ and we first will derive an estimate in $B_r(0)$. Starting again with

$$u(y) - u(0) = \int_0^1 y \cdot (\nabla u)(ry))dr$$

one finds

$$\begin{aligned}
\int_{|y|=s} |u(y) - u(0)| \, d\sigma_y &\leq \int_{|y|=s} \int_0^1 |y \cdot (\nabla u)(ry))| \, dr d\sigma_y \\
&\leq \int_{|y|=s} \int_0^1 s \, |(\nabla u)(ry))| \, dr d\sigma_y \\
&= s^{n-1} \int_{|w|=1} \int_0^s |\nabla u(tw))| \, dt d\sigma_w \\
&= s^{n-1} \int_{|w|=1} \int_0^s \frac{|\nabla u(tw))|}{|tw|^{n-1}} t^{n-1} dt d\sigma_w \\
&= s^{n-1} \int_{|y|<s} \frac{|\nabla u(y))|}{|y|^{n-1}} dy \leq s^{n-1} \int_{|y|<r} \frac{|\nabla u(y))|}{|y|^{n-1}} dy
\end{aligned}$$

and an integration with respect to s gives:

$$\int_{|y|<r} |u(y) - u(0)| \, dy \leq \frac{1}{n} r^n \int_{|y|<r} \frac{|\nabla u(y))|}{|y|^{n-1}} dy. \tag{2.15}$$

So with ω_n is the surface area of the unit ball in $\mathbb{R}^n$

$$\begin{aligned}
\frac{\omega_n}{n} r^n u(0) &\leq \int_{|y|<r} |u(y) - u(0)| \, dy + \int_{|y|<r} |u(y)| \, dy && (2.16) \\
&\leq \frac{1}{n} r^n \int_{|y|<r} \frac{|\nabla u(y))|}{|y|^{n-1}} dy + \|u\|_{L^1(B_r(0))} \\
&\leq \frac{1}{n} r^n \|\nabla u\|_{L^p(B_r(0))} \left\| |\cdot|^{1-n} \right\|_{L^{\frac{p}{p-1}}(B_r(0))} + && (2.17) \\
&\quad + r^{n-\frac{n}{p}} \left(\frac{\omega_n}{n} \right)^{1-\frac{1}{p}} \|u\|_{L^p(B_r(0))}
\end{aligned}$$

For $p > n$ one finds

$$\left\| |\cdot|^{1-n} \right\|_{L^{\frac{p}{p-1}}(B_r(0))} = \left(\omega_n \int_0^r s^{(1-n)\frac{p}{p-1}+n-1} ds \right)^{\frac{p-1}{p}} = \left(\omega_n \frac{1}{1-\frac{n-1}{p-1}} \right)^{\frac{p-1}{p}} r^{\frac{p-n}{p}}.$$

Since 0 is an arbitrary point and taking $r = 1$ we may conclude that there is $C = C_{n,p}$ such that for $p > n$

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

A bound for the Hölder-part of the norm remains to be proven. Let x and z be two points and fix $m = \frac{1}{2}x + \frac{1}{2}z$ and $r = \frac{1}{2}|x - z|$. As in (2.16) and using $B_r(m) \subset B_{2r}(x) \cap B_{2r}(z)$:

$$\begin{aligned} \frac{\omega_n}{n} r^n |u(x) - u(z)| &= \int_{|y-m|<r} |u(x) - u(z)| \, dy \leq \\ \leq & \int_{|y-m|<r} |u(y) - u(x)| \, dy + \int_{|y-m|<r} |u(y) - u(z)| \, dy \\ \leq & \int_{|y-x|<2r} |u(y) - u(x)| \, dy + \int_{|y-z|<2r} |u(y) - u(z)| \, dy \\ \leq & \frac{1}{n} 2^n r^n \left(\int_{|y-x|<2r} \frac{|\nabla u(y))|}{|y-x|^{n-1}} dy + \int_{|y-z|<2r} \frac{|\nabla u(y))|}{|y-z|^{n-1}} dy \right) \end{aligned}$$

and reasoning as in (2.17)

$$|u(x) - u(z)| \leq C_{n,p} r^{\frac{p-n}{p}} \|\nabla u\|_{L^p(B_r(0))}.$$

So

$$\frac{|u(x) - u(z)|}{|x-z|^{\alpha}} \leq 2^{\alpha} C_{n,p} r^{1-\frac{n}{p}-\alpha} \|\nabla u\|_{L^p(B_r(0))},$$

which is bounded when $x \to z$ if $\alpha < 1 - \frac{n}{p}$. ■

Week 3

Some new and old solution methods I

3.1 Direct methods in the calculus of variations

Whenever the problem is formulated in terms of an energy or some other quantity that should be minimized one could try to skip the derivation of the Euler-Lagrange equation. Instead of trying to solve the corresponding boundary value problem one could try to minimize the functional directly.

Let E be the functional that we want to minimize. In order to do so we need the following:

A. There is some open bounded set K of functions and numbers $E_0 < E_1$ such that:

(a) the functional on K is bounded from below by E_0 :

$$\text{for all } u \in K : E(u) \geq E_0,$$

(b) on the boundary of K the functional is bounded from below by E_1:

$$\text{for all } u \in \partial K : E(u) \geq E_1,$$

(c) somewhere inside K the functional has a value in between:

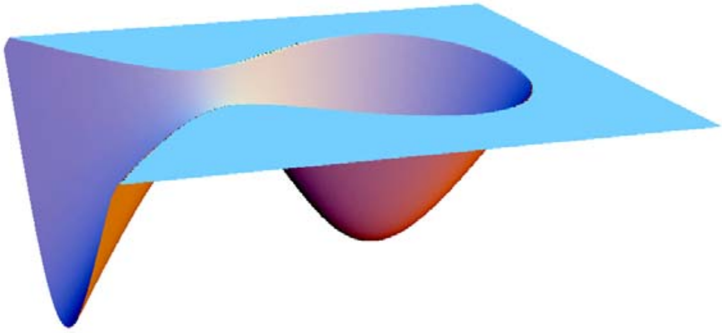
$$\text{there is } u_0 \in K : E(u_0) < E_1.$$

Then $\tilde{E} := \inf_{u \in K} E(u)$ exists and hence we may assume that there is a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset K$, that is, a sequence such that

$$E(u_1) \geq E(u_2) \geq E(u_3) \geq \cdots \geq \tilde{E} \text{ and } \lim_{n \to \infty} E(u_n) = \tilde{E}.$$

This does not yet imply that there is a function $\tilde{u}$ such that $E(\tilde{u}) = \inf_{u \in K} E(u)$.

Definition 3.1.1 *A function $\tilde{u} \in K$ such that $E(\tilde{u}) = \inf_{u \in K} E(u)$ is called a minimizer for E on K.*



Sometimes there is just a local minimizer.

B. In order that u_n leads us to a minimizer the following three properties would be helpful:

(a) the function space should be a Banach-space that fits with the functional; there should be enough functions such that a minimizer is among them

(b) the sequence $\{u_n\}_{n=1}^{\infty}$ converges or at least has a convergent subsequence in some topology (compactness of the set K), and

(c) if a sequence $\{u_n\}_{n=1}^{\infty}$ converges to u_∞ in this topology, then it should also hold that $\lim_{n\to\infty} E(u_n) = E(u_\infty)$ or, which is sufficient: $\lim_{n\to\infty} E(u_n) \geq E(u_\infty)$.

After this heuristic introduction let us try to give some sufficient conditions.

First let us remark that the properties mentioned under A all follow from coercivity.

Definition 3.1.2 *Let $(X, \|\cdot\|)$ be a function space and suppose that $E : X \to \mathbb{R}$ is such that for some $f \in C\left(\mathbb{R}_0^+; \mathbb{R}\right)$ with $\lim\limits_{t\to\infty} f(t) = \infty$ it holds that*

$$E(u) \geq f(\|u\|), \tag{3.1}$$

then E is called coercive.

So let $(X, \|\cdot\|)$ be a function space such that (3.1) holds. Notice that this condition implies that we cannot use a norm with higher order derivatives than the ones appearing in the functional.

Now let us see how the conditions in A follow. The function f has a lower bound which we may use as E_0. Next take any function $u_0 \in X$ and take any $E_1 > E(u_0)$. Since $\lim_{t\to\infty} f(t) = \infty$ we may take M such that $f(t) \geq E_1$ for all $t \geq M$ and set $K = \{u \in X; \|u\| < M\}$. The assumptions in A are satisfied.

Now let us assume that $(X, \|\cdot\|)$ is some Sobolev space $W^{m,p}(\Omega)$ with $1 < p < \infty$ and that

$$K = \left\{ u \in W^{m,p}(\Omega); \text{'boundary conditions' and } \|u\|_{W^{m,p}(\Omega)} < M \right\}.$$

Obviously $W^{m,p}(\Omega)$ is not finite dimensional so that bounded sets are not precompact. So the minimizing sequence in K that we have is, although bounded, has no reason to be convergent in the norm of $W^{m,p}(\Omega)$. The result that saves us is the following.

Theorem 3.1.3 *A bounded sequence $\{u_k\}_{k=1}^{\infty}$ in a reflexive Banach space has a weakly convergent subsequence.*

N.B. The sequence u_k converges weakly to u in the space X, in symbols $u_k \rightharpoonup u$ in X, means $\phi(u_k) \to \phi(u)$ for all $\phi \in X'$, the bounded linear functionals on X. This theorem can be found for example in H. Brezis: Analyse Fonctionnelle (Théorème III.27). And yes, the Sobolev spaces such as $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ with $1 < p < \infty$ are reflexive.

So although the minimizing sequence $\{u_n\}_{n=1}^{\infty}$ is not strongly convergent it has at least a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ that weakly converges. Let us call u_∞ this weak limit in $W^{m,p}(\Omega)$. So

$$u_{n_k} \rightharpoonup u_\infty \text{ weakly in } W^{m,p}(\Omega). \tag{3.2}$$

For the Sobolev space $W^{m,p}(\Omega)$ the statement in (3.2) coincides with

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} u_{n_k} \rightharpoonup \left(\frac{\partial}{\partial x}\right)^{\alpha} u_\infty$$

weakly in $L^p(\Omega)$ for all multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq m$ and $\left(\frac{\partial}{\partial x}\right)^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}$.

The final touch comes from the assumption that E is 'sequentially weakly lower semi-continuous'.

Definition 3.1.4 *Let $(X, \|\cdot\|)$ be a Banach space. The functional $E : X \to \mathbb{R}$ is called sequentially weakly lower semi-continuous, if*

$$u_k \rightharpoonup u \textit{ weakly in } X$$

implies

$$E(u) \leq \liminf_{n\to\infty} E(u_n).$$

We will collect the result discussed above in a theorem. Indeed, the coercivity condition gives us an open bounded set of functions K, numbers E_0, E_1, a function $u_0 \in K^o$ and hence a minimizing sequence in K. Theorem 3.1.3 supplies us with a weak limit $u_\infty \in K$ of a subsequence of the minimizing sequence which is also a minimizing sequence itself (so $\inf_{u\in K} E(u) \leq E(u_\infty)$). And the sequentially weakly lower semi-continuity yields $E(u_\infty) \leq \inf_{u\in K} E(u)$.

Theorem 3.1.5 *Let E be a functional on $W^{m,p}(\Omega)$ (or $W_0^{m,p}(\Omega)$) with $1 < p < \infty$ which is:*

1. *coercive;*

2. *sequentially weakly lower semicontinuous.*

Then there exists a minimizer u of E.

Remark 3.1.6 *These 2 conditions are sufficient but neither of them are necessary and most of the time too restrictive. Often one is only concerned with a minimizing a functional E on some subset of X. If you are interested in variational methods please have a look at Chapter 8 in Evans' book or any other book concerned with direct methods in the Calculus of Variations.*

Example 3.1.7 *Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose we want to minimize $E(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - f\ u\right) dx$ for functions u that satisfy $u = 0$ on $\partial\Omega$. The candidate for the reflexive Banach space is $W_0^{1,2}(\Omega)$ with the norm $\|\cdot\|$ defined by*

$$\|u\|_{W_0^{1,2}} := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}.$$

Indeed this is a norm by Poincaré's inequality: there is $C \in \mathbb{R}^+$ such that

$$\int_\Omega u^2 dx \le C \int_\Omega |\nabla u|^2 \, dx$$

and hence

$$\tfrac{1}{\sqrt{1+C}} \|u\|_{W^{1,2}} \le \|u\|_{W_0^{1,2}} \le \|u\|_{W^{1,2}}.$$

Since $W_0^{1,2}(\Omega)$ with $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \ dx$ is even a Hilbert space we may identify $\left(W_0^{1,2}(\Omega)\right)'$ and $W_0^{1,2}(\Omega)$.

The functional E is coercive: using the inequality of Cauchy-Schwarz

$$\begin{aligned}
E(u) &= \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 - f\ u\right) dx \\
&\ge \tfrac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega f^2 dx\right)^{\frac{1}{2}} \\
&\ge \tfrac{1}{2}\|u\|^2_{W_0^{1,2}} - \|u\|_{W_0^{1,2}} \|f\|_{L^2} \\
&\ge \tfrac{1}{2}\|u\|^2_{W_0^{1,2}} - \left(\tfrac{1}{4}\|u\|^2_{W_0^{1,2}} + \|f\|^2_{L^2}\right) \\
&\ge \tfrac{1}{4}\|u\|^2_{W_0^{1,2}} - \|f\|^2_{L^2}.
\end{aligned}$$

The sequentially weakly lower semi-continuity goes as follows. If $u_n \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$, then

$$E(u_n) - E(u) = \int_\Omega \left(\tfrac{1}{2}|\nabla u_n|^2 - \tfrac{1}{2}|\nabla u|^2 - f(u_n - u)\right) dx$$

$$\begin{aligned}
&= \int_\Omega \left(\tfrac{1}{2}|\nabla u_n - \nabla u|^2 + (\nabla u_n - \nabla u)\cdot\nabla u - f(u_n - u)\right) dx \\
&\ge \int_\Omega ((\nabla u_n - \nabla u)\cdot \nabla u - f(u_n - u))\, dx \to 0.
\end{aligned}$$

Here we also used that $\left(v \mapsto \int_\Omega fv dx\right) \in \left(W_0^{1,2}(\Omega)\right)'$. Hence the minimizer exists.

In this example we have used zero Dirichlet boundary conditions which allowed us to use the well-known function space $W_0^{1,2}(\Omega)$. For nonzero boundary conditions a way out is to consider $W_0^{1,p}(\Omega)+g$ where g is an arbitrary $W^{1,p}(\Omega)$-function that satisfies the boundary conditions.

3.2 Solutions in flavours

In Example 3.1.7 we have seen that we found a minimizer $u \in W_0^{1,2}(\Omega)$ and since it is the minimizer of a differentiable functional it follows that this minimizer satisfies

$$\int (\nabla u \cdot \nabla \eta - f\ \eta)\, dx = 0 \text{ for all } \eta \in W_0^{1,2}(\Omega).$$

If we knew that $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ an integration by parts would show

$$\int (-\Delta u - f)\, \eta dx = 0 \text{ for all } \eta \in W_0^{1,2}(\Omega)$$

and hence $-\Delta u = f$ in $L^2(\Omega)$-sense.

Let us fix for the moment the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Definition 3.2.1 *If $u \in W_0^{1,2}(\Omega)$ is such that*

$$\int (\nabla u \cdot \nabla \eta - f\ \eta)\, dx = 0 \textit{ for all } \eta \in W_0^{1,2}(\Omega)$$

holds, then u is called a weak solution of (3.3).

Definition 3.2.2 *If $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ is such that $-\Delta u = f$ in $L^2(\Omega)$-sense, then u is called a strong solution of (3.3).*

Remark 3.2.3 *If a strong solution even satisfies $u \in C^2(\bar{\Omega})$ and hence the equations in (3.3) hold pointwise u is called a classical solution.*

Often one obtains a weak solution and then one has to do some work in order to show that the solution has more regularity properties, that is, it is a solution in the strong sense, a classical solution or even a C^∞-solution. In the remainder we will consider an example that shows such upgrading is not an automated process.

Example 3.2.4 *Suppose that we want to minimize*

$$E(y) = \int_0^1 \left((1 - (y'(x))^2 \right)^2 dx$$

with $y(0) = y(1) = 0$. The appropriate Sobolev space should be $W_0^{1,4}(0,1)$ and indeed E is coercive:

$$\begin{aligned} E(y) &= \int_0^1 \left(1 - 2(y')^2 + (y')^4\right) dx \\ &\geq \int_0^1 \left(-1 + \tfrac{1}{2}(y')^4\right) dx = \tfrac{1}{2} \|y\|_{W_0^{1,4}}^4 - 1. \end{aligned}$$

Here we used $2a^2 \leq \frac{1}{2}a^4 + 2$ (indeed $0 \leq (\varepsilon^{-\frac{1}{2}}a - \varepsilon^{\frac{1}{2}}b)^2$ implies $2ab \leq \varepsilon^{-1}a^2 + \varepsilon b^2$). We will skip the sequentially weakly lower semicontinuity and just assume that *there is a minimizer u in $W_0^{1,4}(0,1)$. For such a minimizer we'll find through $\frac{\partial}{\partial\tau}E\left(y+\tau\eta\right)_{|\tau=0}=0$ that*

$$\int_0^1 \left(-4y' + 4\left(y'\right)^3\right)\eta'\ dx = 0 \text{ for all } \eta \in W_0^{1,4}(0,1).$$

Assuming that such a minimizer is a strong solution, $y \in W^{2,4}(0,1)$ we may integrate by part to find the Euler-Lagrange equation:

$$4\left(1 - 3\left(y'\right)^2\right)y'' = 0.$$

So $y'' = 0$ or $\left(y'\right)^2 = \frac{1}{3}$. A closer inspection gives $y(x) = ax + b$ and plugging in the boundary conditions $y(0) = y(1) = 0$ we come to $y(x) = 0$.

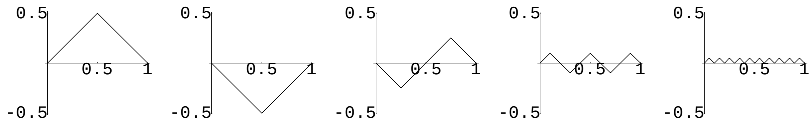
Next let us compute some energies:

$$E(0) = \int_0^1 (1-0)^2\, dx = 1,$$

and $E(y)$ for a special function y in the next exercise.

Exercise 36 *Let E be as in the previous example and compute $E(y)$ for $y(x) = \frac{\sin\pi x}{\pi}$. If you did get that $E\left(y\right) = \frac{3}{8} < 1 = E(0)$ then 0 is not the minimizer. What is wrong here?*

Notice that $E\left(y\right) \geq 0$ for any $y \in W_0^{1,4}(0,1)$, so if we would find a function y such that $E\left(y\right) = 0$ we would have a minimizer. Here are the graphs of a few candidates:



Exercise 37 *Check that the function y defined by $y(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$ is indeed in $W_0^{1,4}(0,1)$.*
And why is y not in $W^{2,4}(0,1)$? For those unfamiliar with derivatives in a weaker sense see the following remark.

Remark 3.2.5 *If a function y is in $C^2\left(0,1\right)$ then we know y' and y'' pointwise in $(0,1)$ and we compute the integral $\left\|y^{(i)}\right\|_{L^4(0,1)}$, $i \in \{0,1,2\}$ in order to find whether or not $y \in W^{2,4}(0,1)$. But if y is not in $C^2(0,1)$ we cannot directly compute $\int_0^1 \left(y''\right)^4 dx$. So how is y'' defined?*

Remember that for differentiable functions y and ϕ an integration by parts show that

$$\int_0^1 y'\phi\ dx = -\int_0^1 y\ \phi'\ dx.$$

One uses this result to define derivatives in the sense of distributions. The distributions that we consider are bounded linear operators from $C_0^\infty\left(\mathbb{R}^n\right)$ to $\mathbb{R}$. Remember that we wrote $C_0^\infty\left(\mathbb{R}^n\right)$ for the infinitely differentiable functions with compact support in $\mathbb{R}^n$.

Definition 3.2.6 *If y is some function defined on $\mathbb{R}^n$ then $\left(\frac{\partial}{\partial x}\right)^\alpha y$ in $C_0^\infty(\mathbb{R}^n)'$ is the distribution V such that*

$$V(\phi) = (-1)^{|\alpha|} \int_{\mathbb{R}^n} y(x) \left(\frac{\partial}{\partial x}\right)^\alpha \phi(x)\ dx.$$

Example 3.2.7 *The derivative of the signum function is twice the Dirac-delta function. This Dirac-delta function is not a true function but a distribution: $\delta(\phi) = \phi(0)$ for all $\phi \in C_0(\mathbb{R})$.*

The signum function is defined by

$$sign(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

We find, calling 'sign$'$ = V', that for ϕ with compact support:

$$\begin{aligned} V(\phi) &= -\int_{-\infty}^{\infty} sign(x)\ \phi'(x)\ dx = \\ &= \int_{-\infty}^{0} \phi'(x)\ dx - \int_{0}^{\infty} \phi'(x)\ dx = 2\phi(0) = 2\delta(\phi). \end{aligned}$$

Exercise 38 *A functional for the energy that belongs to a rectangular plate lying on three supporting walls, clamped on one side and being pushed down by some weight is*

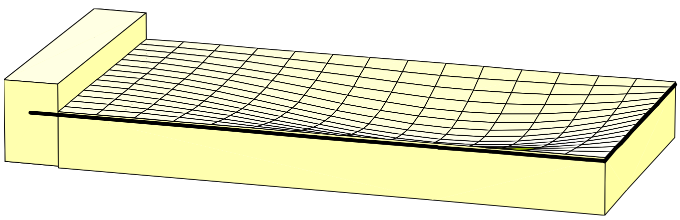
$$E(u) = \int_R \left(\tfrac{1}{2}(\Delta u)^2 - f\ u\right) dxdy.$$

Here $R = (0,\ell) \times (0,b)$ and supported means the deviation u satisfies

$$u(x,0) = u(x,b) = u(\ell,y) = 0,$$

and clamped means

$$u(0,y) = \frac{\partial}{\partial x} u(0,y) = 0.$$

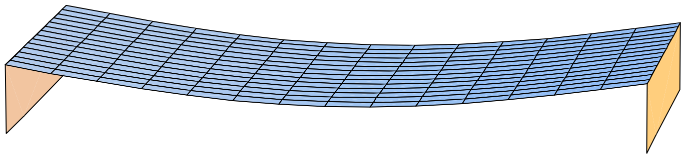


Compute the boundary value problem that comes out.

Exercise 39 *Let us consider the following functional for the energy of a rectangular plate lying on two supporting walls and being pushed down by some weight is*

$$E(u) = \int_R \left(\tfrac{1}{2}(\Delta u)^2 - f\ u\right) dxdy.$$

Here $R = (0,\ell) \times (0,b)$ and supported means the deviation u satisfies $u(0,y) = u(\ell,y) = 0$.



Compute the boundary value problem that comes out.

Exercise 40 *Consider both for Exercise 38 and for Exercise 39 the variational problem*

$$\begin{array}{rcl} E & : & W_{bc}(R) := \overline{\left\{u \in C^{\infty}\left(\bar{R}\right);\ +\text{ boundary conditions}\right\}}^{\|\cdot\|_{W(R)}} \to \mathbb{R} \\ & & \text{with} \quad \|u\|_{W(R)} = \|u\|_{L^2(R)} + \|\Delta u\|_{L^2(R)} . \end{array}$$

For the source term we assume $f \in L^2(R)$.

1. *Is the functional E coercive?*
2. *Is E is sequentially weakly lower semi-continuous?*

Exercise 41 *Show that the equation for Exercise 39 can be written as a system of two second order problems, one for u and one for $v = -\Delta u$. For those who have seen Hopf's boundary Point Lemma: can this system be solved for positive f?*

Exercise 42 *Suppose that we replace both in Exercise 38 and 39 the energy by*

$$E(u) = \int_R \left(\tfrac{1}{2}(\Delta u)^2 + \delta \tfrac{1}{2} |\nabla u|^2 - f\ u\right) dxdy.$$

Compute the boundary conditions for both problems. Can you comment on the changes that appear?

Exercise 43 *What about uniqueness for the problem in Exercises 38 and 39?*

Exercise 44 *Is it true that $W_{br}(R) \subset W^{2,2}(R)$ both for Exercise 38 and for Exercise 39?*

Exercise 45 *Let $u \in W^{4,2}(R) \cap W_{br}(R)$ be a solution for Exercise 39. Show that for all nonzero $\eta \in W_{bc}(R)$ the second variation $\left(\frac{\partial}{\partial \tau}\right)^2 E(u + \tau\eta)_{\tau=0}$ is positive but not necessarily strictly positive.*

In the last exercises we came up with a weak solution, that is $u \in W^{2,2}_{bc}(R)$ satisfying

$$\int_R (\Delta u \Delta \eta - f\eta)\, dxdy = 0 \text{ for all } \eta \in W^{2,2}_{bc}(R) .$$

Starting from such a weak solution one may try to use 'regularity results' in order to find that the u satisfying the weak formulation is in fact is a strong solution: for $f \in L^2(R)$ it holds that $u \in W^{4,2}(R) \cap W^{2,2}_{bc}(R)$.

In the case that f has a stronger regularity, say $f \in C^{\alpha}(R)$, it may even be possible that the solution satisfies $u \in C^{4,\alpha}(R)$. Although since the problems above are linear one might expect that such regularity results are standard available in the literature. On the contrary, these are by no means easy to find. Especially higher order problems and corners form a research area by themselves. A good starting point would be the publications of P. Grisvard.

3.3 Preliminaries for Cauchy-Kowalevski

3.3.1 Ordinary differential equations

One equation

One of the classical ways of finding solutions for an o.d.e. is by trying to find solutions in the form of a power series. The advantage of such an approach is that the computation of the Taylor series is almost automatical. For the n^{th}-order o.d.e.

$$y^{(n)}(x) = f(x, y(x), \ldots, y^{(n-1)}(x)) \tag{3.4}$$

with initial values $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$ the higher order coefficients follow by differentiation and filling in the values already known:

$$\begin{aligned}
y^{(n)}(x_0) &= y_n := f(x_0, y_0, \ldots y_{n-1}) \\
y^{(n+1)}(x_0) &= y_{n+1} := \frac{\partial f}{\partial x}(x_0, y_0, \ldots y_{n-1}) + \sum_{k=0}^{n-1} y_{k+1} \frac{\partial f}{\partial y_i}(x_0, y_0, \ldots y_{n-1}) \\
& \quad \textit{etc.}
\end{aligned}$$

It is well known that such an approach has severe shortcomings. First of all the problem itself needs to fit the form of a power series: f needs to be a real analytic function. Secondly, a power series usually has a rather restricted area of convergence. But if f is appropriate, and if we can show the convergence, we find a solution by its Taylor series

$$y\,(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} y^{(k)}(x_0). \tag{3.5}$$

Remark 3.3.1 *For those who are not to familiar with ordinary differential equations. A sufficient condition for existence and uniqueness of a solution for (3.4) with prescribed initial conditions is $(x, p) \mapsto f(x, p)$ being continuous and $p \mapsto f\,(x, p)$ Lipschitz-continuous.*

Multiple equations

A similar result holds for systems of o.d.e.:

$$\mathbf{y}^{(n)}(x) = \mathbf{f}(x, \mathbf{y}(x), \ldots, \mathbf{y}^{(n-1)}(x)) \tag{3.6}$$

where $\mathbf{y} : \mathbb{R} \to \mathbb{R}^m$ and $\mathbf{f}$ is a vectorfunction. The computation of the coefficients will be more involved but one may convince oneself that given the (vector)values of $\mathbf{y}(x_0)$, $\mathbf{y}'(x_0)$ up to $\mathbf{y}^{(n-1)}(x_0)$ the higher values will follow as before and the Taylor series in (3.5) will be the same with $\mathbf{y}$ instead of y. To have at most one solution by the Taylor series near $x = x_0$ one needs to fix $\mathbf{y}(x_0), \ldots, \mathbf{y}^{(n-1)}(x_0)$.

From higher order to first order autonomous

The trick to go from higher order to first order is well known. Setting $\mathbf{u} = (y, y', \dots, y^{(n-1)})$ the n^{th}-order equation in (3.4) or system in (3.6) changes in

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \mathbf{u} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x, \mathbf{u}) \end{pmatrix} \text{ with } \mathbf{u}(x_0) = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

Finally a simpel addition will make the system autonomous. Just add the equation $u_{n+1}'(x) = 1$ with $u_{n+1}(x_0) = x_0$. We find $u_{n+1}(x) = x$ and by replacing x by u_{n+1} in the o.d.e. or in the system of o.d.e.'s the system becomes autonomous.

So we have found an initial value problem for an autonomous first order system

$$\begin{cases} \mathbf{u}'(x) = \mathbf{g}(\mathbf{u}(x)), \\ \mathbf{u}(x_0) = \mathbf{u}_0. \end{cases} \tag{3.7}$$

Remark 3.3.2 *If we include some parameter dependance in (3.7) we would find the problem*

$$\begin{cases} \mathbf{u}'(x) = \mathbf{g}(\mathbf{u}(x), \mathbf{p}), \\ \mathbf{u}(x_0) = \mathbf{u}_0(\mathbf{p}). \end{cases} \tag{3.8}$$

Such a problem would give a solution of the type

$$\mathbf{u}(x, \mathbf{p}) = \sum_{i=0}^{\infty} \frac{\mathbf{u}^{(k)}(x_0, \mathbf{p})}{k!} (x - x_0)^k$$

and the $\mathbf{u}^{(k)}(x_0, \mathbf{p})$ *itself could be power series in* $\mathbf{p}$ *if the dependance of* $\mathbf{p}$ *in (3.8) would allow. One could think of* $\mathbf{p}$ *as the remaining coordinates in* $\mathbb{R}^n$*. Note that* $\mathbf{u}_0(\mathbf{p})$ *would prescribe the values for* $\mathbf{u}$ *on a hypersurface of dimension* $n - 1$.

Exercise 46 *Compute* $\mathbf{u}^{(k)}(x_0)$ *from (3.7) for* $k = 0, 1, 2, 3$ *in terms of* g *and the initial conditions.*

3.3.2 Partial differential equations

The idea of Cauchy-Kowalevski could be phrased as: let us compute the solution of the p.d.e. by a Taylor-series. The complicating factor becomes that it is not so clear what initial values one should impose.

Taylor series with multiple variables

Before we consider partial differential equations let us recall the Taylor series in multiple dimensions. For an analytic function $v : \mathbb{R}^n \to \mathbb{R}$ it reads as

$$v(x) = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n}^{|\alpha|=k} \frac{\left(\frac{\partial}{\partial x}\right)^\alpha v(x_0)}{\alpha!} (x - x_0)^\alpha$$

where

$$\begin{aligned}
\alpha! &= \alpha_1!\alpha_2!\dots\alpha_n!, \\
\left(\frac{\partial}{\partial x}\right)^{\alpha} &= \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \\
(x-x_0)^{\alpha} &= (x_1-x_{1,0})^{\alpha_1}(x_2-x_{2,0})^{\alpha_2}\dots(x_n-x_{n,0})^{\alpha_n}.
\end{aligned}$$

The $\alpha \in \mathbb{N}^n$ that appears here is called a multiindex.

Exercise 47 *Take your favourite function f of two variables and compute the Taylor polynomial of degree 3 in $(0,0)$:*

$$\sum_{k=0}^{3}\sum_{\alpha\in\mathbb{N}^2}^{|\alpha|=k}\frac{\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}f\right)(0,0)}{\alpha!}x^{\alpha}.$$

Several flavours of nonlinearities

A linear partial differential equation is of order m if it is of the form

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|\leq m} A_\alpha(x)\left(\frac{\partial}{\partial x}\right)^{\alpha} u\,(x) = f\,(x)\,, \tag{3.9}$$

and the $a_\alpha(x)$ with $|\alpha| = m$ should not disappear.

Every p.d.e. that cannot be written as in (3.9) is not linear (= nonlinear?). Nevertheless one sometimes makes a distinction.

Definition 3.3.3 *1. A m^{th} order semilinear p.d.e. is as follows:*

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=m} A_\alpha(x)\left(\frac{\partial}{\partial x}\right)^{\alpha} u\,(x) = f\left(x, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{m-1}u}{\partial x_n^{m-1}}\right).$$

2. A m^{th} order quasilinear p.d.e. is as follows:

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=m} A_\alpha(x, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{m-1}u}{\partial x_n^{m-1}})\left(\frac{\partial}{\partial x}\right)^{\alpha} u\,(x) = f\left(x, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{m-1}u}{\partial x_n^{m-1}}\right).$$

3. The remaining nonlinear equations are the genuine nonlinear ones.

Example 3.3.4 *Here are some nonlinear equations:*

- *The Korteweg-de Vries equation:* $u_t + uu_x + u_{xxx} = 0$.
- *The minimal surface equation:* $\nabla \cdot \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0$.
- *The Monge-Ampère equation:* $u_{xx}u_{yy} - u_{xy}^2 = f$.

One could call them respectively semilinear, quasilinear and 'strictly' nonlinear.

3.3.3 The Cauchy problem

As we have seen for o.d.e.'s an m^{th}-order ordinary differential equation needs m initial conditions in order to have a unique solution. A standard set of such initial conditions consists of giving y, y', $\ldots$, $y^{(m-1)}$ a fixed value. The analogon of such an 'initial value' problem for an m^{th}-order partial differential equation in $\mathbb{R}^n$ would be to describe all derivatives up to order $m-1$ on some $(n-1)$-dimensional manifold. This is a bit too much. Indeed if one prescribes y on a smooth manifold $\mathcal{M}$, then also all tangential derivatives are determined. Similar, if $\frac{\partial}{\partial \nu} y$, the normal derivative on $\mathcal{M}$ is given then also all tangential derivatives of $\frac{\partial}{\partial \nu} y$ are determined. Etcetera. So it will be sufficient to fix all normal derivatives from order 0 up to order $m-1$.

Definition 3.3.5 *The Cauchy problem for the m^{th}-order partial differential equation $F\left(x, y, \frac{\partial}{\partial x_1} y, \ldots, \frac{\partial^m}{\partial x_n^m} y\right) = 0$ on $\mathbb{R}^n$ with respect to the smooth $(n-1)$-dimensional manifold $\mathcal{M}$ is the following:*

$$\left\{ \begin{array}{ll} F\left(x, y, \frac{\partial}{\partial x_1} y, \ldots, \frac{\partial^m}{\partial x_n^m} y\right) = 0 & \text{in } \mathbb{R}^n, \\ \qquad y = \phi_0 & \\ \qquad \frac{\partial}{\partial \nu} y = \phi_1 & \\ \qquad \vdots & \text{on } \mathcal{M}, \\ \qquad \frac{\partial^{m-1}}{\partial \nu^{m-1}} y = \phi_{m-1} & \end{array} \right. \tag{3.10}$$

where ν is the normal on $\mathcal{M}$ and where the ϕ_i are given.

Exercise 48 *Consider $\mathcal{M} = \{(x_1, x_2);\ x_1^2 + x_2^2 = 1\}$ with $\phi_0(x_1, x_2) = x_1$ and $\phi_1(x_1, x_2) = x_2$. Let ν be the outside normal direction on $\mathcal{M}$. Suppose u is a solution of $\frac{\partial^2}{\partial x_1^2} u + \frac{\partial^2}{\partial x_2^2} u = 1$ in a neighborhood of $\mathcal{M}$, say for $1 - \varepsilon < x_1^2 + x_2^2 < 1 + \varepsilon$ that satisfies*

$$\left\{ \begin{array}{ll} u = \phi_0 & \text{for } x \in \mathcal{M}, \\ \frac{\partial}{\partial \nu} u = \phi_1 & \text{for } x \in \mathcal{M}. \end{array} \right.$$

Compute $\frac{\partial^2}{\partial \nu^2} u$ on $\mathcal{M}$ for such a solution.

Exercise 49 *The same question but with the differential equation replaced by $\frac{\partial^2}{\partial x_1^2} u - \frac{\partial^2}{\partial x_2^2} u = 1$.*

3.4 Characteristics I

So the Cauchy problem consists of an m^{th}-order partial differential equation for say u in (a subset of) $\mathbb{R}^n$, an $(n-1)$-dimensional manifold $\mathcal{M}$, the normal direction ν on $\mathcal{M}$ and with u, $\frac{\partial}{\partial \nu}u$, ... $\frac{\partial^{n-1}}{\partial \nu^{n-1}}u$ given on $\mathcal{M}$. One may guess that the differential equation should be of order m in the ν-direction.

Example 3.4.1 *Consider $\frac{\partial}{\partial x_1}u + \frac{\partial}{\partial x_2}u = g(x_1, x_2)$ for a given function g. This is a first order p.d.e. but if we take new coordinates $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$, and set $\tilde{g}(x_1+x_2, x_1-x_2) = g(x_1, x_2)$ and $\tilde{u}(x_1+x_2, x_1-x_2) = u(x_1, x_2)$, we find*

$$\frac{\partial}{\partial x_1}u + \frac{\partial}{\partial x_2}u = 2\frac{\partial}{\partial y_1}\tilde{u}.$$

The y_2-derivative dissappeared. So we have to solve the o.d.e.

$$2\frac{\partial}{\partial y_1}\tilde{u} = \tilde{g}. \tag{3.11}$$

If for example $u(x_1, 0) = \varphi(x_1)$ is given for $x_1 \in \mathbb{R}$, then

$$\tilde{u}(y_1, y_2) = \tilde{u}(x_1, x_1) + \tfrac{1}{2}\int_{x_1}^{y_1} \tilde{g}(s, y_2)ds,$$

which can be rewritten to a formula for u.

But if $u(x_1, x_1) = \varphi(x_1)$ is given for $x_1 \in \mathbb{R}$, then $\tilde{u}(y_1, 0) = \varphi(\frac{1}{2}y_1)$ and this is in general not compatible with (3.11). One finds that the solution is constant on each line $y_2 = c$. That means we are not allowed to prescribe the function u on such a line and if we want the Cauchy-problem to be well-posed for a 1-d-manifold (curve) we better take a curve with no tangential in that direction.

The line (or curve) along which the p.d.e. takes the form of an o.d.e. is called a characteristic curve.

In higher dimensions and for higher order equations the notion of characteristic manifold becomes somewhat involved. For quasilinear differential equations a characteristic curve or manifold in general will even depend on the solution itself. So we will use a more usefull concept that is found in the book of Evans. Instead of giving a definition of characteristic curve or characteristic manifold we will give a definition of noncharacteristic manifolds.

Definition 3.4.2 *A manifold $\mathcal{M}$ is called noncharacteristic for the differential equation*

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=m} A_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha u(x) = B\left(x, u, \frac{\partial}{\partial x_1}u, \dots, \frac{\partial^m}{\partial x_n^m}u\right). \tag{3.12}$$

if for $x \in \mathcal{M}$ and every normal direction ν to $\mathcal{M}$ in x:

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=m} A_\alpha(x)\nu^\alpha \neq 0. \tag{3.13}$$

We say ν is a singular direction for (3.12) if $\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=m} A_\alpha(x)\nu^\alpha = 0$.

Remark 3.4.3 *The condition in (3.13) means that the coefficient of the differential equation of the m^{th}-order in the ν-direction does not disappear. In case that (3.12) would be an o.d.e. in the ν direction we would 'loose' one order in the differential equation.*

Remark 3.4.4 *Sobolev defines a surface given by $F(x_1, \ldots, x_n) = 0$ to be characteristic if the (2^{nd} order) differential equation, on changing from the variables $x_1, \ldots, x_n$ to new variables $y_1 = F(x_1, \ldots, x_n), y_2, \ldots, y_n$ with y_2 to y_n arbitrary functions of x_1 to x_n, such that all the y_i are continuous and have first order derivatives and a non-zero Jacobian in a neighbourhood of the surface under consideration, it happens that the coefficient $\bar{A}_{11}$ of $\frac{\partial^2}{\partial y_1^2}$ vanishes on this surface.*
The definition of noncharacteristic gives a condition for each point on the manifold. Also Sobolev's definition of characteristic does. So not characteristic and noncharacteristic are not identical.

3.4.1 First order p.d.e.

Consider the first order semilinear partial differential equation

$$\sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} y(x) = B(x, y). \tag{3.14}$$

A solution is a function $y : \mathbb{R}^n \to \mathbb{R}$. The Cauchy problem for such a first order p.d.e. consist of prescribing initial data on a (smooth) $n-1$-dimensional surface. We will see that we cannot choose just any surface. If we consider a curve $t \mapsto x(t)$ that satisfies $x'(t) = A(x(t))$ then one finds for a solution y on this line that

$$\begin{aligned} \frac{\partial}{\partial t} y(x(t)) &= \sum_{i=1}^{n} x_i'(t) \frac{\partial}{\partial x_i} y(x) = \\ &= \sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} y(x) = B(x(t), y(x(t))) \end{aligned}$$

and hence that we may solve y on this curve when one value $y(x_0)$ is given. Indeed

$$\begin{cases} x'(t) = A(x(t)) \\ x(0) = x_0 \end{cases} \tag{3.15}$$

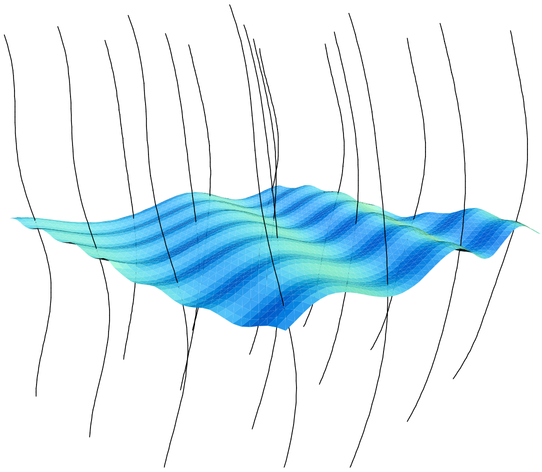
is an o.d.e. system with a unique solution for analytic A (Lipschitz is sufficient).

Definition 3.4.5 *The solution $t \mapsto x(t)$ is called a characteristic curve for (3.14).*

With this curve $t \mapsto x(t)$ known one obtains a second o.d.e. for $t \mapsto y(x(t))$. Set $Y(t) = y(x(t))$ and Y solves

$$\begin{cases} Y'(t) = B(x(t), Y(t)) \\ Y(0) = y(x_0) \end{cases} \tag{3.16}$$

So if we would prescribe the initial values for the solution on an $n-1$-dimensional surface we should take care that this surface does not intersect each of those characteristic curves more than once. This is best guaranteed by assuming that the characteristic directions are not tangential to the surface. For a first order this means that $x'(t)$ should not be tangential to this surface. In other words $A\left(x(t)\right)\cdot\nu = x'(t)\cdot\nu \neq 0$ where ν is the normal direction of the surface.



Characteristic lines and an appropriate surface with prescribed values

Exercise 50 *Show that also for the first order quasilinear equations*

$$\sum_{i=1}^{m} A_i(x,y)\frac{\partial}{\partial x_i}y(x) = B\left(x,y\right)$$

one obtains a system of o.d.e.'s for the curve $t\mapsto x(t)$ and the solution $t\mapsto y\left(x(t)\right)$ along this curve.

3.4.2 Classification of second order p.d.e. in two dimensions

The standard form of a second order semilinear partial differential equation in two dimensions is

$$a\ u_{x_1x_1} + 2b\ u_{x_1x_2} + c\ u_{x_2x_2} = f\left(x,u,\nabla u\right). \tag{3.17}$$

Here f is a given function and one is searching for a function u.

We may write this second order differential operator as follows:

$$a\ \frac{\partial^2}{\partial x_1^2} + 2b\ \frac{\partial^2}{\partial x_1 \partial x_2} + c\ \frac{\partial^2}{\partial x_2^2} = \nabla\cdot\begin{pmatrix} a & b \\ b & c \end{pmatrix}\nabla$$

and since the matrix is symmetric we may diagonalize it by an orthogonal matrix T:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = T^T\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}T.$$

Definition 3.4.6 *The equation in (3.17) is called*

- *elliptic if* λ_1 *and* λ_2 *have the same (nonzero) sign;*
- *hyperbolic if* λ_1 *and* λ_2 *have opposite (nonzero) signs;*
- *parabolic if* $\lambda_1 = 0$ *or* $\lambda_2 = 0$.

Remark 3.4.7 *In the case that* a, b *and* c *do depend on* x *one says that the equation is elliptic, hyperbolic respectively parabolic in* x *if the eigenvalues* $\lambda_1(x)$ *and* $\lambda_2(x)$ *are as above (freeze the coefficients in* x*). Note that*

$$\begin{aligned} &\nabla \cdot \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} \nabla u = \\ = \quad & a(x)\ u_{x_1x_1} + 2b(x)\ u_{x_1x_2} + c(x)\ u_{x_2x_2} + \gamma(x)u_{x_1} + \delta(x)u_{x_2} \end{aligned}$$

where

$$\gamma = \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_2} \text{ and } \delta = \frac{\partial b}{\partial x_1} + \frac{\partial c}{\partial x_2}.$$

So the highest order doesn't change.

For the constant coefficients case one reduce the highest order coefficients to standard form by taking new coordinates $Tx = y$. One finds that (3.17) changes to

$$\lambda_1\ u_{y_1y_1} + \lambda_2\ u_{y_2y_2} = \tilde{f}\,(y, u, \nabla u)\,.$$

Depending on the signs of λ_1 and λ_2 we may introduce a scaling $y_i = \sqrt{|\lambda_i|}z_i$ and come up with either one of the following three possibilities:

- a semilinear Laplace equation: $u_{z_1z_1} + u_{z_2z_2} = \hat{f}(z, u, \nabla u)$;
- a semilinear wave equation: $u_{z_1z_1} - u_{z_2z_2} = \hat{f}(z, u, \nabla u)$;
- $u_{z_1z_1} = \hat{f}(z, u, \nabla u)$, which can be turned into a heat equation when $\hat{f}(z, u, \nabla u) = c_1u_{z_1} + c_2u_{z_2}$.

Exercise 51 *Show that (3.17) is*

1. *elliptic if and only if* $\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0$;
2. *parabolic if and only if* $\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} = 0$;
3. *hyperbolic if and only if* $\det \begin{pmatrix} a & b \\ b & c \end{pmatrix} < 0$;

Exercise 52 *Show that under any regular change of coordinates* $y = Bx$, *with* B *a nonsingular* 2×2*-matrix, the classification of the differential equation in (3.17) does not change.*

Definition 3.4.8 *For a linear p.d.e. operator* $Lu := \sum_{\alpha\in\mathbb{N}^n}^{|\alpha|\leq m} A_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha u$ *one introduces the symbol by replacing the derivation* $\frac{\partial}{\partial x_i}$ *by* ξ_i :

$$\mathit{symbol}_L(\xi) = \sum_{\alpha\in\mathbb{N}^n}^{|\alpha|\leq m} A_\alpha(x)\ \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}.$$

Exercise 53 *Show that for second order operators with constant coefficients in two dimensions:*

1. *The solutions of* $\{\xi \in \mathbb{R}^2;\ symbol_L(\xi) = k\}$ *consists of an ellips for some* $k \in \mathbb{R}$*, if and only if* L *is elliptic.*

2. *The solutions of* $\{\xi \in \mathbb{R}^2;\ symbol_L(\xi) = k\}$ *consists of a hyperbola for some* $k \in \mathbb{R}$*, if and only if* L *is hyperbolic.*

3. *If the solutions of* $\{\xi \in \mathbb{R}^2;\ symbol_L(\xi) = k\}$ *consists of a parabola for some* $k \in \mathbb{R}$*, then* L *is parabolic.*

4. *If the solutions of* $\{\xi \in \mathbb{R}^2;\ symbol_L(\xi) = k\}$ *do not fit the descriptions above for any* $k \in \mathbb{R}$*, then* L *is not a real p.d.e.: it can be transformed into an o.d.e..*

3.5 Characteristics II

3.5.1 Second order p.d.e. revisited

Let us consider the three standard equations that we obtained separately.

1. $u_{x_1x_1} + u_{x_2x_2} = f(x, u, \nabla u)$. We may take any line ℓ through a point $\bar{x}$ and a orthonormal coordinate system with ν perpendicular and τ tangential to find in the new coordinates $u_{\nu\nu} + u_{\tau\tau} = \tilde{f}(y, u, u_\tau, u_\nu)$. So with $x = y_1\nu + y_2\tau$

$$\begin{cases} u_{y_2y_2} = -u_{y_1y_1} + \tilde{f}(y, u, u_{y_1}, u_{y_2}) & \text{for } y \in \mathbb{R}^2, \\ u(y_1, 0) = \phi(y_1) & \text{for } y_1 \in \mathbb{R}, \\ u_{y_2}(y_1, 0) = \psi(y_1) & \text{for } y_1 \in \mathbb{R}, \end{cases} \tag{3.18}$$

and find that the coefficients in the power series are defined. Indeed if we prescribe $u(y_1, 0)$ and $u_{y_2}(y_1, 0)$ for y_1 we also know

$$\left(\frac{\partial}{\partial y_1}\right)^k u(y_1, 0) \text{ and } \left(\frac{\partial}{\partial y_1}\right)^k \frac{\partial}{\partial y_2} u(y_1, 0) \text{ for all } k \in \mathbb{N}^+. \tag{3.19}$$

Using the differential equation and (3.19) we find

$$\left(\frac{\partial}{\partial y_2}\right)^2 u(y_1, 0) \text{ and hence } \left(\frac{\partial}{\partial y_1}\right)^k \left(\frac{\partial}{\partial y_2}\right)^2 u(y_1, 0) \text{ for all } k \in \mathbb{N}^+. \tag{3.20}$$

Differentiating the differential equation with respect to y_2 and using the results from (3.19-3.20) gives

$$\left(\frac{\partial}{\partial y_2}\right)^3 u(y_1, 0) \text{ and hence } \left(\frac{\partial}{\partial y_1}\right)^k \left(\frac{\partial}{\partial y_2}\right)^3 u(y_1, 0) \text{ for all } k \in \mathbb{N}^+. \tag{3.21}$$

By repeating these steps we will find all Taylor-coefficients and, hoping the series converges, the solution.

Going back through the transformation one finds that for any p.d.e. as in (3.17) that is elliptic the Cauchy-problem looks well-posed.(We would like to say '**is well-posed**' but since we didn't state the corresponding theorem ...)

For elliptic p.d.e. any manifold is noncharacteristic.

2. $u_{x_1x_1} - u_{x_2x_2} = f(x, u, \nabla u)$. Taking new orthogonal coordinates by

$$y = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} x \tag{3.22}$$

the differential operator turns into

$$\left(\cos^2\alpha - \sin^2\alpha\right) \frac{\partial^2}{\partial y_1^2} - 4\cos\alpha\sin\alpha \frac{\partial^2}{\partial y_1 \partial y_2} + \left(\sin^2\alpha - \cos^2\alpha\right) \frac{\partial^2}{\partial y_2^2}$$

and the equation we may write as

$$u_{y_2y_2} = u_{y_1y_1} + \frac{4\cos\alpha\sin\alpha}{\sin^2\alpha - \cos^2\alpha} u_{y_1y_2} + \hat{f}(y,u,\nabla u)$$

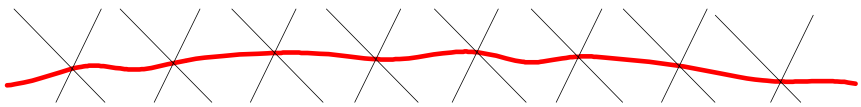
if and only if $\sin^2\alpha \neq \cos^2\alpha$. If we suppose that $\sin^2\alpha \neq \cos^2\alpha$ and prescribe $u(y_1,0)$ and $u_{y_2}(y_1,0)$ then we may proceed as for case 1 and find the coefficients of the Taylor series. On the other hand, if $\sin^2\alpha = \cos^2\alpha$, then we are left with

$$2u_{y_1y_2} = \pm\tilde{f}(y,u,u_{y_1},u_{y_2}).$$

Prescribing just $u(y_1,0)$ is not sufficient since then $u_{y_2}(y_1,0)$ is still unknown but prescribing both $u(y_1,0)$ and $u_{y_2}(y_1,0)$ is too much since now $u_{y_1y_2}(y_1,0)$ follows from $u_{y_2}(y_1,0)$ directly and from the differential equation indirectly and unless some special relation between f and u_{y_2} holds two different values will come out. So the Cauchy problem

$$\begin{cases} u_{x_2x_2} = u_{x_1x_1} + f(x,u,u_{x_1},u_{x_2}) & \text{for } x \in \mathbb{R}^2, \\ u(x) = \phi(x) & \text{for } x \in \ell, \\ u_\nu(x) = \psi(x) & \text{for } x \in \ell, \end{cases}$$

is (probably) only well-posed for ℓ not parallel to $\binom{1}{1}$ or $\binom{1}{-1}$. Going back through the transformation one finds that any p.d.e. as in (3.17) that is hyperbolic has at each point two directions for which the Cauchy problem has a singularity if one of these two direction coincides with the manifold $\mathcal{M}$. For any other manifold (a curve in 2 dimensions) the Cauchy-problem is (could be) well-posed.

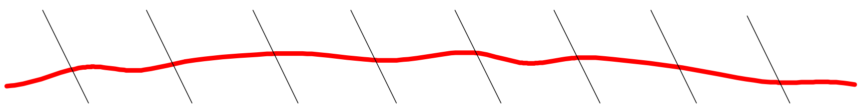


2^{nd} *order hyperbolic in 2d: in each point there are two singular directions.*

3. $u_{x_1x_1} = f(x,u,\nabla u)$. Doing the transformation as in (3.22) one finds

$$\sin^2\alpha\ u_{y_2y_2} = 2\sin\alpha\cos\alpha\ u_{y_1y_2} - \cos^2\alpha\ u_{y_1y_1} + \hat{f}(y,u,\nabla u).$$

which a (probably) well-posed Cauchy problem (3.18) except for α a multiple of π. The lines $x_2 = c$ give the sigular directions for this equation. Going back through the transformation one finds that any p.d.e. as in (3.17) that is parabolic has one family of directions where the highest order derivatives disappear. For any 1-d manifold that crosses one of those lines exactly once, the Cauchy-problem is (as we will see) well-posed.



2^{nd} *order parabolic in 2d: in each point there is one singular direction.*

Exercise 54 *Consider the one-dimensional wave equation (a p.d.e. in* $\mathbb{R}^2$*)* $u_{tt} = c^2u_{xx}$ *for some* $c > 0$.

1. *Write the Cauchy problem for* $u_{tt} = c^2u_{xx}$ *on* $\mathcal{M} = \{(x,t); t = 0\}$.

2. *Use* $u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial t}\right)$ *to show that solutions of this one-dimensional wave equation can be written as*

$$u(x,t) = f(x-ct) + g(x+ct).$$

3. *Now compute the* f *and* g *such that* u *is a solution of the Cauchy problem with general* φ_0 *and* φ_1.

The explicit formula for this Cauchy problem that comes out is named after d'Alembert. If you promise to make the exercise before continuing reading, this is it:

$$u(x,t) = \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\varphi_0(x-ct) + \tfrac{1}{2}\int_{x-ct}^{x+ct}\varphi_1(s)ds.$$

Exercise 55 *Try the find a solution for the Cauchy problem for* $u_{tt}+u_t = c^2 u_{xx}$ *on* $\mathcal{M} = \{(x,t); t=0\}$.

3.5.2 Semilinear second order in higher dimensions

That means we are considering differential equations of the form

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=2} A_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha u = f(x,u,\nabla u) \tag{3.23}$$

and we want to prescribe u and the normal derivative u_ν on some analytic $n-1$-dimensional manifold. Let us suppose that this manifold is given by the implicit equation $\Phi(x) = c$. So the normal in $\bar{x}$ is given by

$$\nu = \frac{\nabla\Phi(\bar{x})}{|\nabla\Phi(\bar{x})|}.$$

In order to have a well-defined Cauchy problem we should have a nonvanishing $\frac{\partial^2}{\partial\nu^2}u$ component in (3.23).

Going to new coordinates We want to rewrite (3.23) such that we recognize the $\frac{\partial^2}{\partial\nu^2}u$ component. We may do so by filling up the coordinate system with tangential directions $\tau_1,\dots,\tau_{n-1}$ at $\bar{x}$. Denoting the old coordinates by $\{x_1,x_2,\dots,x_n\}$ and the new coordinates by $\{y_1,y_2,\dots,y_n\}$ we have

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \nu_1 & \tau_{1,1} & \cdots & \tau_{n-1,1} \\ \nu_2 & \tau_{1,2} & \cdots & \tau_{n-1,n} \\ \vdots & \vdots & & \vdots \\ \nu_n & \tau_{1,n} & \cdots & \tau_{n-1,n} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and since this transformation matrix T is orthonormal one finds

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_n \\ \tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,n} \\ \vdots & \vdots & & \vdots \\ \tau_{n-1,1} & \tau_{n-1,n} & \cdots & \tau_{n-1,n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Hence

$$\frac{\partial}{\partial x_i} = \frac{\partial y_1}{\partial x_i}\frac{\partial}{\partial y_1} + \sum_{j=1}^{n-1}\frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} = \nu_i \frac{\partial}{\partial y_1} + \sum_{j=1}^{n-1}\tau_{i,j}\frac{\partial}{\partial y_j}$$

So (3.23) turns into

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=2} A_\alpha(x)\prod_{i=1}^{n}\left(\nu_i\frac{\partial}{\partial y_1} + \sum_{j=1}^{n-1}\tau_{i,j}\frac{\partial}{\partial y_j}\right)^{\alpha_i} u = f\left(x, u, \nabla u\right) \tag{3.24}$$

and the factor in front of the $\frac{\partial^2}{\partial y_1^2}u$ component, which is our notation for $\frac{\partial^2}{\partial \nu^2}u$ at $\bar{x}$ in (3.23), is

$$\sum_{\alpha\in\mathbb{N}^n}^{|\alpha|=2} A_\alpha(\bar{x})\nu^\alpha = \nu\cdot\begin{pmatrix} A_{2,0,\dots,0}(\bar{x}) & \frac{1}{2}A_{1,1,\dots,0}(\bar{x}) & \cdots & \frac{1}{2}A_{1,0,\dots,1}(\bar{x}) \\ \frac{1}{2}A_{1,1,\dots,0}(\bar{x}) & A_{0,2,\dots,0}(\bar{x}) & \cdots & \frac{1}{2}A_{0,1,\dots,1}(\bar{x}) \\ \vdots & \vdots & & \vdots \\ \frac{1}{2}A_{1,0,\dots,1}(\bar{x}) & \frac{1}{2}A_{0,1,\dots,1}(\bar{x}) & \cdots & A_{0,0,\dots,2}(\bar{x}) \end{pmatrix}\nu. \tag{3.25}$$

Classification by the number of positive, negative and zero eigenvalues

Let us call the matrix in (3.25) M. Since it is symmetric we may diagonalize it and find a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal. Notice that this matrix does not depend on ν. Also remember that we have a well-posed Cauchy problem if $\nu \cdot M\nu \neq 0$.

Let P be the number of positive eigenvalues and N the number of negative ones. There are three possibilities.

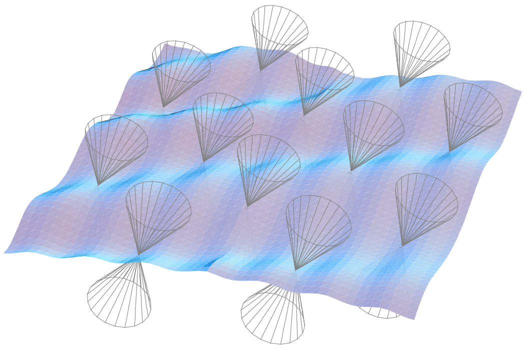
1. $(P, N) = (n, 0)$ of $(P, N) = (0, n)$. Then all eigenvalues of M have the same sign and in that case $\nu \cdot M\nu \neq 0$ for any ν with $|\nu| = 1$. So any direction of the normal ν is fine and there is no direction that a manifold may not contain. Equation (3.23) is called elliptic.

2. $P + N = n$ but $1 \leq P \leq n-1$. There are negative and positive eigenvalues but no zero eigenvalues. Then there are directions ν such that $\nu \cdot M\nu = 0$. Equation (3.23) is called hyperbolic.

3. $P + N < n$. There are eigenvalues equal to 0. Equation (3.23) is called parabolic.

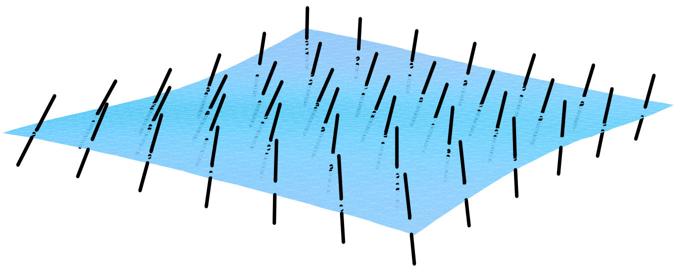
There exists a more precise subclassification according to the number of positive, negative and zero eigenvalues.

In dimension 3 with $(N, P) = (1, 2)$ or $(2, 1)$ one finds a cone of singular directions near $\bar{x}$.



The singular directions at $\bar{x}$ form a cone in the hyperbolic case.

In dimension 3 with $(N, P) = (0, 2)$ or $(0, 2)$ one finds one singular direction near $\bar{x}$.



Singular directions in a parabolic case.

3.5.3 Higher order p.d.e.

The classification for higher order p.d.e. in so far it concerns parabolic and hyperbolic becomes a bit of a zoo. Elliptic however can be formulated by the absence of any singular directions. For an m^{th}-order semilinear equation as

$$\sum_{|\alpha|=m} A_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha u = F\left(x, u, \tfrac{\partial}{\partial x_1}u, \dots, \left(\frac{\partial}{\partial x}\right)^\alpha u\right) \tag{3.26}$$

it can be rephrased as:

Definition 3.5.1 *The differential equation in (3.26) is called elliptic in $\bar{x}$ if*

$$\sum_{|\alpha|=m} A_\alpha(\bar{x})\xi^\alpha \neq 0 \text{ for all } \xi \in \mathbb{R}^n \backslash \{0\}.$$

Exercise 56 *Show that for an elliptic m^{th}-order semilinear equation (as in (3.26)) with (as always) real coefficients $A_\alpha(x)$ the order m is even.*

Week 4

Some old and new solution methods II

4.1 Cauchy-Kowalevski

4.1.1 Statement of the theorem

Let us try to formulate some of the 'impressions' we might have got from the previous sections.

Heuristics 4.1.1

1. *For a well-posed Cauchy problem of a general p.d.e. one should have a manifold $\mathcal{M}$ to which the singular directions of the p.d.e. are strictly nontangential.*

2. *The condition for the singular directions of the p.d.e.*

$$\sum_{|\alpha|=m} A_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha u + F(x,u,\frac{\partial}{\partial x_1}u,\dots,\frac{\partial^{m-1}}{\partial x_n^{m-1}}u) = 0$$

is $\sum_{|\alpha|=m} A_\alpha(x)\,\nu^\alpha = 0$. A noncharacteristic manifold $\mathcal{M}$ means that if ν is normal to $\mathcal{M}$ then $\sum_{|\alpha|=m} A_\alpha(x)\,\nu^\alpha \neq 0$.

3. *The amount of singular directions increases from elliptic (none) via parabolic to hyperbolic.*

4. *(Real-valued) elliptic p.d.e.'s have an even order m; so odd order p.d.e.'s have singular directions.*

Now for the theorem.

Theorem 4.1.2 (Cauchy-Kowalevski for one p.d.e.) *We consider the Cauchy problem for the quasilinear p.d.e.*

$$\sum_{|\alpha|=m} A_\alpha\left(x,u,\tfrac{\partial}{\partial x_1}u,\dots,\tfrac{\partial^{m-1}}{\partial x_n^{m-1}}u\right)\left(\tfrac{\partial}{\partial x}\right)^\alpha u = F\left(x,u,\tfrac{\partial}{\partial x_1}u,\dots,\tfrac{\partial^{m-1}}{\partial x_n^{m-1}}u\right)$$

with the boundary conditions as in (3.10) with an $(n-1)$-dimensional manifold $\mathcal{M}$ and given functions $\varphi_0, \varphi_1, \dots \varphi_{m-1}$.

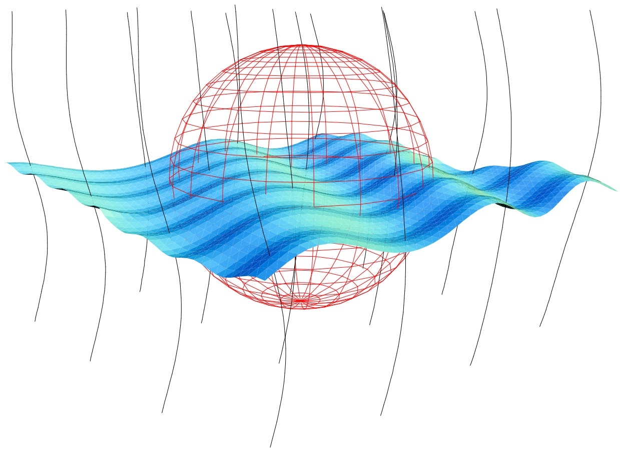
Assume that

1. *$\mathcal{M}$ is a real analytic manifold with $\bar{x} \in \mathcal{M}$. Let ν be the normal direction to $\mathcal{M}$ in $\bar{x}$.*

2. *$\varphi_0, \varphi_1, \dots \varphi_{m-1}$ are real analytic functions defined in a neighborhood of $\bar{x}$.*

3. *$(x, \mathbf{p}) \mapsto A_a(x, \mathbf{p})$ and $(x, \mathbf{p}) \mapsto F(x, \mathbf{p})$ are real analytic in a neighborhood of $\bar{x}$.*

If $\sum_{|\alpha|=m} A_\alpha \left(\bar{x}, u, \frac{\partial}{\partial x_1} u, \dots, \frac{\partial^{m-1}}{\partial x_n^{m-1}} u\right) \ \nu^\alpha \neq 0$ with the derivatives of u replaced by the appropriate (derivatives of) φ_i at $x = \bar{x}$, then there is a ball $B_r(\bar{x})$ and a unique analytic solution u of the Cauchy-problem in this ball.



For Cauchy-Kowalevski the characteristic curves should not be tangential to the manifold.

The theorem above seems quite general but in fact it is not. There is a version of the Theorem of Cauchy-Kowalevski for systems of p.d.e. But similar as for o.d.e. we may reduce Theorem 4.1.2 or the version for systems to a version for a system of first-order p.d.e.

4.1.2 Reduction to standard form

We are considering

$$\sum_{|\alpha|=m} A_\alpha \left(\tfrac{\partial}{\partial x}\right)^\alpha u = F \tag{4.1}$$

where A_α and F may depend on x and the derivatives $\left(\frac{\partial}{\partial x}\right)^\beta u$ of orders $|\beta| \leq m-1$. In fact the u could even be a vector function, that is $u : \mathbb{R}^n \to \mathbb{R}^N$, and the A_α would be $N \times N$ matrices. Are you still following? Then one might recognize that not only solutions turn into vector-valued solutions but also that characteristic curves turn into characteristic manifolds.

For the moment let us start with the single equation in (4.1), which is written out:

$$\sum_{|\alpha|=m} A_\alpha \left(x, u, \tfrac{\partial}{\partial x_1}u, \ldots, \tfrac{\partial^{m-1}}{\partial x_n^{m-1}}u\right) \left(\tfrac{\partial}{\partial x}\right)^\alpha u = F\left(x, u, \tfrac{\partial}{\partial x_1}u, \ldots, \tfrac{\partial^{m-1}}{\partial x_n^{m-1}}u\right).$$

The first step is to take new coordinates that fit with the manifold. This we have done before: see Definitions 2.1.8 and 2.2.1 where now C^1 or C^m is replaced by real analytic. In fact we are only considering local solutions so one box/coordinate system is sufficient. Next we introduce new coordinates for which the manifold is flat. The assumption that the manifold $\mathcal{M}$ does not contain singular directions is preserved in this new coordinate system. Let us write $\{y_1, y_2, \ldots, y_{n-1}, t\}$ for the coordinates where the flattened manifold is $\{t = 0\}$. We are let to the following problem:

$$\hat{A}_{(0,\ldots,0,m)} \left(\tfrac{\partial}{\partial t}\right)^m \hat{u} = \sum_{k=0}^{m-1} \sum_{\beta\in\mathbb{N}^{n-1}}^{|\beta|=m-k} \hat{A}_{(\beta,k)} \left(\tfrac{\partial}{\partial t}\right)^k \left(\tfrac{\partial}{\partial y}\right)^\beta \hat{u} + \hat{F} \tag{4.2}$$

where the coefficients $\hat{A}_\alpha$ and $\hat{F}$ depend on $y, t, \hat{u}, \frac{\partial}{\partial y_1}\hat{u}, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}}\hat{u}$. The assumption that $\mathcal{M}$ does not contain singular directions is transferred to the condition $\hat{A}_{(0,\ldots,0,m)} \neq 0$ and hence we are allowed to divide by this factor to find just $\left(\frac{\partial}{\partial t}\right)^m \hat{u}$ on the left hand side of (4.2). The prescribed Cauchy-data on $\mathcal{M}$ change into

$$\begin{aligned}
&\hat{u} = \hat{\varphi}_0 = \varphi_0 \\
&\tfrac{\partial}{\partial t}\hat{u} = \hat{\varphi}_1 = \varphi_1 + \sum_{i=1}^{n} a_i \tfrac{\partial}{\partial \tau_i}\varphi_0 \\
&\quad\vdots \\
&\tfrac{\partial^{m-1}}{\partial t^{m-1}}\hat{u} = \hat{\varphi}_{m-1} = \varphi_{m-1} + \sum_{k=0}^{m-2} \sum_{\beta\in\mathbb{N}^{m-1}}^{k+|\beta|=m-2} a_{(\beta,k)} \left(\tfrac{\partial}{\partial \tau}\right)^\beta \varphi_k
\end{aligned} \tag{4.3}$$

taking the appropriate identification of $x \in \mathcal{M}$ and $(y,t) \in \mathbb{R}^{n-1} \times \{0\}$.

As a next step one transfers the higher order initial value problem (4.2)-(4.3) into a first order system by setting

$$\tilde{u}_{(\beta,k)} = \left(\tfrac{\partial}{\partial t}\right)^k \left(\tfrac{\partial}{\partial y}\right)^\beta \hat{u} \text{ for } |\beta| + k \leq m-1 \text{ and } \beta \in \mathbb{N}^{n-1}.$$

We find for $k \leq m-1$ that

$$\tfrac{\partial}{\partial t}\tilde{u}_{(0,\ldots,0,k)} = \left(\tfrac{\partial}{\partial t}\right)^{k+1} \hat{u}$$

and if $|\beta| + k \leq m-1$ with $|\beta| \neq 0$, let us say i_β is the first nonzero index with $\beta_i \geq 1$, that

$$\tfrac{\partial}{\partial t}\tilde{u}_{(\beta,k)} = \left(\tfrac{\partial}{\partial y_{i_\beta}}\right) \tilde{u}_{(\tilde{\beta},k+1)} \text{ with } \tilde{\beta} = (0,\ldots,0,\beta_i - 1, \beta_{i+1}, \ldots, \beta_{n-1})$$

So combining these equations we have

$$\left\{\begin{array}{ll}
\frac{\partial}{\partial t}\tilde{u}_{(0,k)} = \tilde{u}_{(0,k+1)} & \text{for } 0 \leq k \leq m-2, \\
\frac{\partial}{\partial t}\tilde{u}_{(\beta,k)} = \left(\frac{\partial}{\partial y_i}\right) \tilde{u}_{(\tilde{\beta},k+1)} & \text{for } 0 \leq k + |\beta| \leq m-1 \text{ with } |\beta| \neq 0, \\
\frac{\partial}{\partial t}\tilde{u}_{(0,\ldots,0,m-1)} = \sum_{k=0}^{m-1} \sum_{\beta\in\mathbb{N}^{n-1}}^{|\beta|=m-k} \tilde{A}_{(\beta,k)} \frac{\partial}{\partial y_{i_\beta}} \tilde{u}_{(\tilde{\beta},k)} + \hat{F} &
\end{array}\right. \tag{4.4}$$

where $\tilde{A}_{(\beta,k)}$ and $\hat{F}$ are analytic functions of $y, t, \tilde{u}_{(\beta,k)}$ with $|\beta| + k \leq m - 1$. The boundary conditions go to

$$\begin{cases} \tilde{u}_{(0,k)} = \hat{\varphi}_k & \text{for } 0 \leq k \leq m-1, \\ \tilde{u}_{(\beta,k)} = \left(\frac{\partial}{\partial y}\right)^{\beta} \hat{\varphi}_k & \text{for } 0 \leq k + |\beta| \leq m-1 \text{ with } |\beta| \neq 0 \end{cases} \tag{4.5}$$

With some extra equations for the coordinates in order to make the system autonomous as we did for the o.d.e. the equation (4.4)-(4.5) can be recasted by vector notation:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{v} = \check{\mathbf{A}}_j(\mathbf{v})\frac{\partial}{\partial y_j}\mathbf{v} + \check{\mathbf{F}}(\mathbf{v}) & \text{in } \mathbb{R}^{n-1} \times \mathbb{R}^+, \\ \mathbf{v}(0,y) = \boldsymbol{\psi}(y) & \text{on } \mathbb{R}^{n-1}. \end{cases}$$

As a final step we consider $\mathbf{u}(t,y) = \mathbf{v}(t,y) - \boldsymbol{\psi}(0)$ in order to reduce to vanishing initial data at the origin:

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{u} = \mathbf{A}_j(\mathbf{u})\frac{\partial}{\partial y_j}\mathbf{u} + \mathbf{F}(\mathbf{u}) & \text{in } \mathbb{R}^{n-1} \times \mathbb{R}^+, \\ \mathbf{u}(0,y) = \boldsymbol{\varphi}(y) & \text{on } \mathbb{R}^{n-1}, \\ \mathbf{u}(0,0) = \mathbf{0}. \end{cases} \tag{4.6}$$

The formulation of the Cauchy-Kowalevski Theorem is usually stated for this system which is more general than the one we started with.

Theorem 4.1.3 (Cauchy-Kowalevski) *Assume that $\mathbf{u} \mapsto \mathbf{A}_j(\mathbf{u})$, $\mathbf{u} \mapsto \mathbf{F}(\mathbf{u})$ and $y \mapsto \boldsymbol{\varphi}(y)$ are real analytic functions in a neighborhood of the origin. Then there exists a unique analytic solution of (4.6) in a neighborhood of the origin.*

For a detailed proof one might consider the book by Fritz John.

4.2 A solution for the 1d heat equation

4.2.1 The heat kernel on $\mathbb{R}$

Let us try to derive a solution for

$$u_t - u_{xx} = 0 \tag{4.7}$$

and preferably one that satisfies some initial condition(s). The idea is to look for similarity solutions, that is, try to find some solution that fits with the scaling found in the equation. Or in other words exploit this scaling to reduce the dimension. For the heat equation one notices that if $u(t,x)$ solves (4.7) also $u\left(c^2t, cx\right)$ is a solution for any $c \in \mathbb{R}$. So let us try the (maybe silly) approach to take c such that $c^2t = 1$. Then it could be possible to get a solution of the form $u(1, t^{-1/2}x)$ which is just a function of one variable. Let's try and set $u(1, t^{-1/2}x) = f(t^{-1/2}x)$. Putting this in (4.7) we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) f(x/\sqrt{t}) = -\frac{x}{2t\sqrt{t}} f'(x/\sqrt{t}) - \frac{1}{t} f''(x/\sqrt{t}).$$

If we set $\xi = x/\sqrt{t}$ we find the o.d.e.

$$\tfrac{1}{2}\xi f'(\xi) + f''(\xi) = 0.$$

This o.d.e. is separable and through

$$\frac{f''(\xi)}{f'(\xi)} = -\tfrac{1}{2}\xi$$

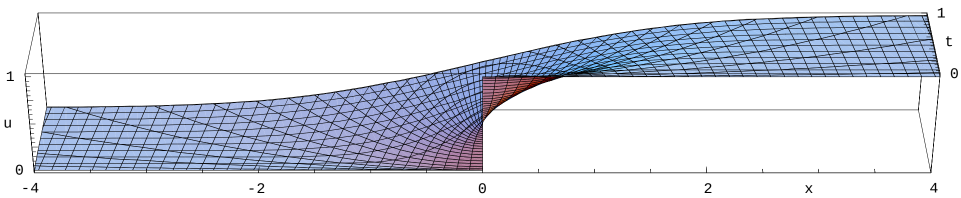
and integrating both sides we obtain first

$$\ln|f'(\xi)| = -\tfrac{1}{4}\xi^2 + C \text{ and } f'(\xi) = c_1 e^{-\frac{1}{4}\xi^2}$$

and next

$$f(x/\sqrt{t}) = c_2 + c_1 \int_{-\infty}^{x/\sqrt{t}} e^{-\frac{1}{4}\xi^2} d\xi.$$

The constant solution is not very interesting so we forget c_2 and for reasons to come we take c_1 such that $c_1 \int_{-\infty}^{\infty} e^{-\frac{1}{4}\xi^2} d\xi = 1$. A computation gives $c_1 = \frac{1}{2}\pi^{-1/2}$. Here is a graph of that function.



One finds that

$$\bar{u}(t,x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{t}} e^{-\frac{1}{4}\xi^2} d\xi$$

is a solution of the equation. Moreover for $(t,x) \neq (0,0)$ (and $t \geq 0$) this function is infinitely differentiable. As a result also all derivatives of this $\bar{u}$ satisfy the heat equation. One of them is special. The function $x \mapsto \bar{u}(0,x)$ equals 0 for $x<0$ and 1 for $x>0$. The derivative $\frac{\partial}{\partial x}\bar{u}(0,x)$ equals Dirac's δ-function in the sense of distributions. Let us give a name to the x-derivative of $\bar{u}$

$$p(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

This function p is known as the heat kernel on $\mathbb{R}$.

Exercise 57 *Let $\varphi \in C_0^\infty(\mathbb{R})$. Show that $z(t,x) = \int_{-\infty}^{\infty} p\,(t, x-y)\,\varphi(y)dy$ is a solution of*

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) z(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ z(0,x) = \varphi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 58 *Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ and set $p_3(t,x) = p(t,x_1)p(t,x_2)p(t,x_3)$. Show that $z(t,x) = \int_{\mathbb{R}^3} p_3\,(t, x-y)\,\varphi(y)dy$ is a solution of*

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) z(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ z(0,x) = \varphi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 59 *Show that the function $w\,(x,t) = \frac{x}{t\sqrt{t}} e^{-\frac{x^2}{4t}}$ is a solution to the heat equation and satisfies*

$$\lim_{t \downarrow 0} w(x,t) = 0 \text{ for any } x \in \mathbb{R}.$$

Is this a candidate for showing that

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) z(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ z(0,x) = \varphi(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

has a non-unique solution in $C^2\left(\mathbb{R}_0^+ \times \mathbb{R}\right)$?

Is the convergence of $\lim_{t\downarrow 0} w(x,t) = 0$ uniform with respect to $x \in \mathbb{R}$?

4.2.2 On bounded intervals by an orthonormal set

For the heat equation in one dimension on a bounded interval, say $(0,1)$, we may find a solution for initial values $\varphi \in L^2$ by the following construction. First let us recall the boundary value problem:

$$\begin{cases} \text{the p.d.e.:} & u_t - u_{xx} = 0 & \text{in } (0,1) \times \mathbb{R}^+, \\ \text{the initial value:} & u = \varphi & \text{on } (0,1) \times \{0\}, \\ \text{the boundary condition:} & u = 0 & \text{on } \{0,1\} \times \mathbb{R}^+. \end{cases} \tag{4.8}$$

We try to solve this boundary value problem by *separation of variables.* That is, we try to find functions of the form $u(t,x) = T(t)X(x)$ that solve the boundary value problem in (4.8) for some initial value.

Putting these special u's into (4.8) we find the following

$$\begin{cases} T'(t)X(x) - T(t)X''(x) = 0 & \text{for } 0<x<1 \text{ and } t>0, \\ T(0)X(x) = \varphi(x) & \text{for } 0<x<1, \\ T(t)X(0) = T(t)X(1) = 0 & \text{for } t>0. \end{cases}$$

The only nontrivial solutions for X are found by

$$\frac{T'(t)}{T(t)} = c = -\frac{X''(x)}{X(x)} \text{ and } X(0) = X(1) = 0,$$

where also the constant c is unknown. Some lengthy computations show that these nontrivial solutions are

$$X_k(x) = \alpha \sin(k\pi x) \text{ and } c = k^2\pi^2.$$

(Without further knowledge one tries $c < 0$, finds functions $ae^{\sqrt{c}x} + be^{-\sqrt{c}x}$ that do not satisfy the boundary conditions except when they are trivial. Similarly $c = 0$ doesn't bring anything. For some $c > 0$ some combination of $\cos(\sqrt{c}x)$ and $\sin(\sqrt{c}x)$ does satisfy the boundary condition.)

Having these X_k we can now solve the corresponding T and find

$$T_k(t) = e^{-k^2\pi^2 t}.$$

So the special solutions we have found are

$$u(t,x) = \alpha\ e^{-k^2\pi^2 t} \sin(k\pi x).$$

Since the problem is linear one may combine such special functions and find that one can solve

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0 & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0,x) = \sum_{k=1}^m \alpha_k \sin(k\pi x) & \text{for } 0 < x < 1, \\ u(t,0) = u(t,1) = 0 & \text{for } t > 0. \end{cases} \tag{4.9}$$

Now one should remember that the set of functions $\{e_k\}_{k=1}^\infty$ with $e_k(x) = \sqrt{2}\sin(k\pi x)$ is a complete orthonormal system in $L^2(0,1)$, so that we can write any function in $L^2(0,1)$ as $\sum_{k=1}^\infty \alpha_k e_k(\cdot)$. Let us recall:

Definition 4.2.1 *Suppose H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The set of functions $\{e_k\}_{k=1}^\infty$ in H is called a complete orthonormal system for H if the following holds:*

1. *$\langle e_k, e_\ell\rangle = \delta_{kl}$ (orthonormality);*
2. *$\lim_{m\to\infty} \left\|\sum_{k=1}^m \langle e_k, f\rangle\ e_k(\cdot) - f(\cdot)\right\|_H = 0$ for all $f \in H$ (completeness).*

Remark 4.2.2 *As a consequence one finds Parseval's identity: for all $f \in H$*

$$\begin{aligned} \|f\|_H^2 &= \lim_{m\to\infty} \left\|\sum_{k=1}^m \langle e_k, f\rangle\ e_k(\cdot)\right\|_H^2 = \\ &= \lim_{m\to\infty} \left\langle \sum_{k=1}^m \langle e_k, f\rangle\ e_k(\cdot), \sum_{j=1}^m \langle e_j, f\rangle\ e_j(\cdot)\right\rangle = \\ &= \lim_{m\to\infty} \sum_{k=1}^m \sum_{j=1}^m \langle e_k, f\rangle \langle e_j, f\rangle \delta_{kj} = \\ &= \lim_{m\to\infty} \sum_{k=1}^m \langle e_k, f\rangle^2 = \|\langle e_k, f\rangle\|_{\ell^2}^2 . \end{aligned}$$

Remark 4.2.3 *Some standard complete orthonormal systems on $L^2(0,1)$, which are useful for the differential equations that we will meet, are:*

- $\left\{\sqrt{2}\sin(k\pi\cdot)\,;k\in\mathbb{N}^+\right\}$;
- $\left\{1,\sqrt{2}\sin(2k\pi\cdot)\,,\sqrt{2}\cos(2k\pi\cdot)\,;k\in\mathbb{N}^+\right\}$;
- $\left\{1,\sqrt{2}\cos(k\pi\cdot)\,;k\in\mathbb{N}\right\}$.

Another famous set of orthonormal functions are the Legendre polynomials:

- $\left\{\sqrt{k+\frac{1}{2}}P_k(\cdot);k\in\mathbb{N}\right\}$ *with* $P_k(x)=\frac{1}{2^k k!}\frac{d^k}{dx^k}\left(x^2-1\right)^k$ *is a complete orthonormal set in* $L^2(-1,1)$.

Remark 4.2.4 *The use of orthogonal polynomials for solving linear p.d.e. is classical. For further reading we refer to the books of Courant-Hilbert, Strauss, Weinberger or Sobolev.*

Exercise 60 *Write down the boundary conditions that the third set of functions in Remark 4.2.3 satisfy.*

Exercise 61 *The same question for the second set of functions.*

Exercise 62 *Find a complete orthonormal set of functions for*

$$\begin{cases} -u''=f \text{ in } (-1,1)\,,\\ u(-1)=0,\\ u(1)=0.\end{cases}$$

Exercise 63 *Find the appropriate set of orthonormal functions for*

$$\begin{cases} -u''=f \text{ in } (0,1)\,,\\ u(0)=0,\\ u_x(1)=0.\end{cases}$$

Exercise 64 *Suppose that $\{e_k;k\in\mathbb{N}\}$ is an orthonormal system in $L^2(0,1)$. Let $f\in L^2(0,1)$. Show that for $a_0,\dots,a_k\in\mathbb{R}$ the following holds:*

$$\left\|f-\sum_{i=0}^{k}\langle f,e_i\rangle\,e_i\right\|_{L^2(0,1)}\le\left\|f-\sum_{i=0}^{k}a_i\;e_i\right\|_{L^2(0,1)}.$$

1. *Show that the set of Legendre polynomials is an orthonormal collection.*
2. *Show that every polynomial $p:\mathbb{R}\to\mathbb{R}$ can be written by a finite combination of Legendre polynomials.*
3. *Prove that the Legendre polynomials form a complete orthonormal set in $L^2(-1,1)$.*

Coming back to the problem in (4.8) we have found for $\varphi\in L^2(0,1)$ the following solution:

$$u(t,x)=\sum_{k=1}^{\infty}\left\langle\sqrt{2}\sin(k\pi\cdot)\,,\varphi\right\rangle\;e^{-\pi^2k^2t}\sqrt{2}\sin(k\pi x)\,. \tag{4.10}$$

Exercise 65 *Let $\varphi \in L^2(0,1)$ and let u be the function in (4.10).*

1. *Show that $x \mapsto u(t,x)$ is in $L^2(0,1)$ for all $t \geq 0$.*
2. *Show that all functions $x \mapsto \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial t^j}u(t,x)$ for $t > 0$ are in $L^2(0,1)$.*
3. *Show that there is $c > 0$ (independent of φ) with $\|u(t,\cdot)\|_{W^{2,2}(0,1)} \leq \frac{c}{t}\|\varphi\|_{L^2(0,1)}$.*

4.3 A solution for the Laplace equation on a square

The boundary value problem that we will consider has zero Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } (0,1)^2, \\ u = 0 & \text{on } \partial\left((0,1)^2\right), \end{cases} \tag{4.11}$$

where $f \in L^2(0,1)^2$. We will try the use a separation of variables approach and first try to find sufficiently many functions of the form $u(x,y) = X(x)Y(y)$ that solve the corresponding eigenvalue problem:

$$\begin{cases} -\Delta \phi = \lambda\phi & \text{in } (0,1)^2, \\ \phi = 0 & \text{on } \partial\left((0,1)^2\right). \end{cases} \tag{4.12}$$

At least there should be enough functions to approximate f in L^2-sense. Putting such functions in (4.12) we obtain

$$\begin{cases} -X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y) & \text{for } 0 < x, y < 1, \\ X(x)Y(0) = 0 = X(x)Y(1) & \text{for } 0 < x < 1, \\ X(0)Y(y) = 0 = X(1)Y(y) & \text{for } 0 < y < 1. \end{cases}$$

Skipping the trivial solution we find

$$\begin{cases} -\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = \lambda & \text{for } 0 < x, y < 1, \\ X(0) = 0 = X(1) \\ Y(0) = 0 = Y(1) \end{cases}$$

Our quest for solutions X, Y, λ leads us after some computations to

$$\begin{aligned} X_k(x) &= \sin(k\pi x), \\ Y_\ell(y) &= \sin(\ell\pi y), \\ \lambda_{k,\ell} &= \pi^2 k^2 + \pi^2 \ell^2. \end{aligned}$$

Since $\left\{\sqrt{2}\sin(k\pi\cdot); k \in \mathbb{N}^+\right\}$ is a complete orthonormal system in $L^2(0,1)$ (and $\left\{\sqrt{2}\sin(\ell\pi\cdot); \ell \in \mathbb{N}^+\right\}$ on $L^2\binom{1}{0}$, sorry $L^2(0,1)$) the combinations supply such a system in $L^2((0,1)^2)$.

Lemma 4.3.1 *If $\{e_k; k \in \mathbb{N}\}$ is a complete orthonormal system in $L^2(0,a)$ and if $\{f_k; k \in \mathbb{N}\}$ is a complete orthonormal system in $L^2(0,b)$, then $\{\tilde{e}_{k,\ell}; k, \ell \in \mathbb{N}\}$ with $\tilde{e}_{k,\ell}(x,y) = e_k(x) f_\ell(y)$ is a complete orthonormal system on $L^2((0,a) \times (0,b))$.*

Since we now have a complete orthonormal system for (4.11) we can approximate $f \in L^2(0,1)^2$. Setting

$$e_{k,l}(x,y) = 2\sin(k\pi x)\sin(\ell\pi y)$$

we have

$$\lim_{M\to\infty} \left\| f - \sum_{m=2}^{M} \sum_{i+j=m} \langle e_{i,j}, f\rangle\, e_{i,j} \right\|_{L^2(0,1)^2} = 0.$$

Since for the right hand side f in (4.11) replaced by $\tilde{f} := \sum_{m=2}^{M} \sum_{i+j=m} \langle e_{i,j}, f \rangle \, e_{i,j}$ the solution is directly computable, namely

$$\tilde{u} = \sum_{m=2}^{M} \sum_{i+j=m} \frac{\langle e_{i,j}, f \rangle}{\lambda_{ij}} e_{i,j},$$

there is some hope that the solution for $f \in L^2(0,1)^2$ is given by

$$u = \lim_{M \to \infty} \sum_{m=2}^{M} \sum_{i+j=m} \frac{\langle e_{i,j}, f \rangle}{\lambda_{ij}} e_{i,j}, \tag{4.13}$$

at least in L^2-sense. Indeed, one can show that for every $f \in L^2(0,1)^2$ a solution u is given by (4.13). This solution will be unique in $W^{2,2}(0,1)^2 \cap W_0^{1,2}(0,1)^2$.

Exercise 66 *Suppose that* $f \in L^2(0,1)^2$.

1. *Show that the following series converge in* L^2*-sense:*

$$\sum_{m=2}^{M} \sum_{i+j=m} i^{\alpha_1} j^{\alpha_2} \frac{\langle e_{i,j}, f \rangle}{\lambda_{ij}} e_{i,j}$$

with $\alpha \in \mathbb{N}^2$ *and* $|\alpha| \leq 2$ *(6 different series).*

2. *Show that* $u(t,x)$, u_x, u_y, u_{xx}, u_{xy} *and* u_{yy} *are in* $L^2(0,1)^2$.

Exercise 67 *Find a complete orthonormal system that fits for*

$$\begin{cases} -\Delta u(x,y) = f(x,y) & \text{for } 0 < x, y < 1, \\ u(0,y) = u(1,y) = 0 & \text{for } 0 < y < 1, \\ u_y(x,0) = u_y(x,1) = 0 & \text{for } 0 < x < 1. \end{cases} \tag{4.14}$$

Exercise 68 *Have a try for the disk in* $\mathbb{R}^2$ *using polar coordinates*

$$\begin{cases} -\Delta u(x) = f(x) & \text{for } |x| < 1, \\ u(x) = 0 & \text{for } |x| = 1. \end{cases} \tag{4.15}$$

In polar coordinates: $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$.

4.4 Solutions for the 1d wave equation

4.4.1 On unbounded intervals

For the 1-dimensional wave equation we have seen a solution by the formula of d'Alembert. Indeed

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \\ u(x,0) = \varphi_0(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

has a solution of the form

$$u(x,t) = \tfrac{1}{2}\varphi_0(x-ct) + \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\int_{x-ct}^{x+ct} \varphi_1(s)ds.$$

We may use this solution to find a solution for the wave equation on the half line:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{for } x \in \mathbb{R}^+ \text{ and } t \in \mathbb{R}^+, \\ u(x,0) = \varphi_0(x) & \text{for } x \in \mathbb{R}^+, \\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \mathbb{R}^+, \\ u(0,t) = 0 & \text{for } t \in \mathbb{R}^+. \end{cases} \tag{4.16}$$

A way to do this is by a 'reflection'. We extend the initial values φ_0 and φ_1 on $\mathbb{R}^-$, use d'Alembert's formula on $\mathbb{R} \times \mathbb{R}_0^+$ and hope that if we restrict the function we have found to $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ a solution of (4.16) comes out.

First the extension. For $i = 1, 2$ we set

$$\tilde{\varphi}_i(x) = \begin{cases} \varphi_i(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\varphi_i(-x) & \text{if } x < 0. \end{cases}$$

Then we make the distinction for

$$\begin{aligned} \mathbf{I}\!: &= \quad \{(x,t)\,; 0 \le t \le cx\}, \\ \mathbf{II}\!: &= \quad \{(x,t)\,; t > c\,|x|\}, \end{aligned}$$

(the case $0 \le t \le c\,|x|$ with $x < 0$ we will not need).

Case **I**. The solution for the modified $\tilde{\varphi}_i$ on $\mathbb{R}$ is

$$\begin{aligned} \tilde{u}\,(x,t) &= \tfrac{1}{2}\tilde{\varphi}_0(x-ct) + \tfrac{1}{2}\tilde{\varphi}_0(x+ct) + \tfrac{1}{2}\int_{x-ct}^{x+ct} \tilde{\varphi}_1(s)ds = \\ &= \tfrac{1}{2}\varphi_0(x-ct) + \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\int_{x-ct}^{x+ct} \varphi_1(s)ds. \end{aligned}$$

Case **II**. The solution for the modified $\tilde{\varphi}_i$ on $\mathbb{R}$ is, now using that $x - ct < 0$

$$\tilde{u}\,(x,t) = \tfrac{1}{2}\tilde{\varphi}_0(x-ct) + \tfrac{1}{2}\tilde{\varphi}_0(x+ct) + \tfrac{1}{2}\int_{x-ct}^{x+ct} \tilde{\varphi}_1(s)ds =$$

$$= \quad -\tfrac{1}{2}\varphi_0(ct-x) + \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\int_{x-ct}^{0} \varphi_1(-s)ds + \tfrac{1}{2}\int_{0}^{x+ct} \varphi_1(s)ds =$$

$$= \quad -\tfrac{1}{2}\varphi_0(ct-x) + \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\int_{ct-x}^{ct+x} \varphi_1(s)ds.$$

Exercise 69 *Suppose that* $\varphi_0 \in C^2(\mathbb{R}_0^+)$ *and* $\varphi_1 \in C^1(\mathbb{R}_0^+)$ *with* $\varphi_0(0) = \varphi_1(0) = 0$.

1. *Show that*
$$u(x,t) = \operatorname{sign}(x-ct)\tfrac{1}{2}\varphi_0(|x-ct|) + \tfrac{1}{2}\varphi_0(x+ct) + \tfrac{1}{2}\int_{|x-ct|}^{x+ct}\varphi_1(s)ds$$
is $C^2(\mathbb{R}_0^+ \times \mathbb{R}_0^+)$ *and a solution to (4.16).*

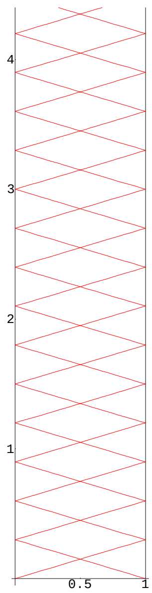
2. *What happens when we remove the condition* $\varphi_1(0) = 0$ *?*

3. *And what if we also remove the condition* $\varphi_0(0) = 0$ *?*

Exercise 70 *Solve the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = \varphi_0(x) & \text{for } x \in \mathbb{R}^+, \\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \mathbb{R}^+, \\ u_x(0,t) = 0 & \text{for } t \in \mathbb{R}^+, \end{cases}$$

using d'Alembert and a reflection.



Exercise 71 *Try to find out what happens when we use a similar approach for*

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{for } (x,t) \in (0,1) \times \mathbb{R}^+, \\ u(x,0) = \varphi_0(x) & \text{for } x \in (0,1), \\ u_t(x,0) = \varphi_1(x) & \text{for } x \in (0,1), \\ u(0,t) = u(1,t) = 0 & \text{for } t \in \mathbb{R}^+. \end{cases}$$

The picture on the right might help.

4.4.2 On a bounded interval

Exercise 72 *We return to the boundary value problem in Exercise 71.*

1. *Use an appropriate complete orthonormal set* $\{e_k(\cdot)\,; k \in \mathbb{N}\}$ *to find a solution:*
$$u(t,x) = \sum_{k=0}^{\infty} a_k(t)\; e_k(x)$$

2. *Suppose that* $\varphi_0, \varphi_1 \in L^2(0,1)$. *Is it true that* $x \mapsto u(t,x) \in L^2(0,1)$ *for all* $t > 0$*?*

3. *Suppose that* $\varphi_0, \varphi_1 \in C[0,1]$. *Is it true that* $x \mapsto u(t,x) \in C[0,1]$ *for all* $t > 0$*?*

4. *Suppose that* $\varphi_0, \varphi_1 \in C_0[0,1]$. *Is it true that* $x \mapsto u(t,x) \in C_0[0,1]$ *for all* $t > 0$*?*

5. *Give a condition on* φ_0, φ_1 *such that* $\sup \left\{ \|u(\cdot, t)\|_{L^2(0,1)} ; t \geq 0 \right\}$ *is bounded.*

6. *Give a condition on* φ_0, φ_1 *such that* $\sup \{|u(x,t)| ; 0 \leq x \leq 1, t \geq 0\}$ *is bounded.*

7. *Give a condition on* φ_0, φ_1 *such that* $\|u(\cdot, \cdot)\|_{L^2((0,1)\times\mathbb{R}^+)}$ *is bounded.*

4.5 A fundamental solution for the Laplace operator

A function u defined on a domain $\Omega \subset \mathbb{R}^n$ are called harmonic whenever

$$\Delta u = 0 \text{ in } \Omega.$$

We will start today by looking for radially symmetric harmonic functions. Since the Laplace-operator Δ in $\mathbb{R}^n$ can be written for radial coordinates with $r = |x|$ by

$$\Delta = r^{1-n} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{LB}$$

where Δ_{LB} is the so-called Laplace-Beltrami operator on the surface of the unit ball $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. So radial solutions of $\Delta u = 0$ satisfy

$$r^{1-n} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} u(r) = 0.$$

Hence $\frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} u(r) = 0$, which implies $r^{n-1} \frac{\partial}{\partial r} u(r) = c$, which implies $\frac{\partial}{\partial r} u(r) = cr^{1-n}$, which implies

$$\begin{array}{ll} \text{for } n = 2: & u(r) = c_1 \log r + c_2, \\ \text{for } n \geq 3: & u(r) = c_1 r^{2-n} + c_2. \end{array}$$

Removing the not so interesting constant c_2 and choosing a special c_1 (the reason will become apparent later) we define

$$\begin{array}{ll} \text{for } n = 2: & F_2(x) = \frac{-1}{2\pi} \log |x|, \\ \text{for } n \geq 3: & F_n(x) = \frac{1}{\omega_n (n-2)} |x|^{2-n}. \end{array} \tag{4.17}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$.

Lemma 4.5.1 *The functions F_n and ∇F_n are in $L^1_{loc}(\mathbb{R}^n)$.*

Next let us recall a consequence of Green's formula:

Lemma 4.5.2 *For $u, v \in C^2(\bar{\Omega})$ and $\Omega \subset \mathbb{R}^n$ a bounded domain with C^1-boundary*

$$\int_\Omega u \ \Delta v \ dy = \int_\Omega \Delta u \ v \ dy + \int_{\partial\Omega} \left(u \ \frac{\partial}{\partial \nu} v - \left(\frac{\partial}{\partial \nu} u \right) v \right) \ d\sigma,$$

where ν is the outwards pointing normal direction.

So if $u, v \in C^2(\bar{\Omega})$ with u is harmonic, we find

$$\int_\Omega u \ \Delta v \ dy = \int_{\partial\Omega} \left(u \ \frac{\partial}{\partial \nu} v - \left(\frac{\partial}{\partial \nu} u \right) v \right) \ d\sigma. \tag{4.18}$$

Now we use this for $v(y) = F_n(x - y)$ with $x \in \mathbb{R}^n$ and the F_n in (4.17). If $x \notin \bar{\Omega}$ then (4.18) holds. If $x \in \Omega$ then since the F_n are not harmonic in 0 we have to drill a hole around x for $y \mapsto F_n(x-y)$. Since the F_n have an integrable singularity we find

$$\int_\Omega F_n(x-y)\ \Delta v(y)\ dy = \lim_{\varepsilon \downarrow 0} \int_{\Omega \backslash B_\varepsilon(x)} F_n(x-y)\ \Delta v(y)\ dy.$$

So instead of Ω we will use $\Omega \backslash B_\varepsilon(x) = \{y \in \Omega; |x-y| > \varepsilon\}$ and find

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{\Omega \backslash B_\varepsilon(x)} F_n(x-y)\ \Delta v(y)\ dy = \\
&= \int_{\partial(\Omega \backslash B_\varepsilon(x))} \left(F_n(x-y)\ \frac{\partial}{\partial \nu} v - \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v \right) d\sigma = \\
&= \int_{\partial \Omega} \left(F_n(x-y)\ \frac{\partial}{\partial \nu} v - \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v \right) d\sigma + \\
&\quad - \lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} \left(F_n(x-y)\ \frac{\partial}{\partial \nu} v - \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v \right) d\sigma. \quad (4.19)
\end{aligned}$$

Notice that the minus sign appeared since outside of $\Omega \backslash B_\varepsilon(x)$ is inside of $B_\varepsilon(x)$. Now let us separately consider the last two terms in (4.19).

We find that

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} F_n(x-y)\ \frac{\partial}{\partial \nu} v\ d\sigma = \\
&= \begin{cases} \text{if } n=2: & \lim_{\varepsilon \downarrow 0} \int_{|\omega|=1} \left(\frac{-1}{2\pi} \log \varepsilon \right)\ \frac{\partial}{\partial \nu} v(x+\varepsilon\omega)\ \varepsilon\ d\omega = 0, \\ \text{if } n \geq 3: & \lim_{\varepsilon \downarrow 0} \int \frac{1}{\omega_n (n-2)} \varepsilon^{2-n}\ \frac{\partial}{\partial \nu} v(x+\varepsilon\omega)\ \varepsilon^{n-1}\ d\omega = 0, \end{cases}
\end{aligned}$$

and since $\frac{\partial}{\partial \nu} F_n(x-y) = -\omega_n^{-1} \varepsilon^{1-n}$ on $\partial B_\varepsilon(x)$ for all $n \geq 2$ it follows that

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{\partial B_\varepsilon(x)} \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v\ d\sigma = \\
&= \lim_{\varepsilon \downarrow 0} \int_{|\omega|=1} -\omega_n^{-1} \varepsilon^{1-n}\ v(x+\varepsilon\omega)\ \varepsilon^{n-1}\ d\omega = -v(x).
\end{aligned}$$

Combining these results we have found, assuming that $x \in \Omega$:

$$\begin{aligned}
v(x) &= \int_\Omega F_n(x-y)\ \left(-\Delta v(y) \right)\ dy + \\
&\quad + \int_{\partial\Omega} F_n(x-y)\ \frac{\partial}{\partial \nu} v\ d\sigma - \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v\ d\sigma.
\end{aligned}$$

If $x \notin \bar{\Omega}$ then we may directly use (4.18) to find

$$\begin{aligned}
0 &= \int_\Omega F_n(x-y)\ \left(-\Delta v(y) \right)\ dy + \\
&\quad + \int_{\partial\Omega} F_n(x-y)\ \frac{\partial}{\partial \nu} v\ d\sigma - \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v\ d\sigma.
\end{aligned}$$

Example 4.5.3 *Suppose that we want to find a solution for*

$$\begin{cases} -\Delta v = f & \text{on } \mathbb{R}^+ \times \mathbb{R}^{n-1}, \\ v = \varphi & \text{on } \{0\} \times \mathbb{R}^{n-1}, \end{cases}$$

say for $f \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1})$ *and* $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$. *Introducing*

$$x^* = (-x_1, x_2, \ldots, x_n)$$

we find that for $x \in \mathbb{R}^+ \times \mathbb{R}^{n-1}$:

$$\begin{aligned} v(x) &= \int_\Omega F_n(x-y) \;\; (-\Delta v(y)) \;\; dy + \\ &\quad + \int_{\partial\Omega} F_n(x-y) \;\; \frac{\partial}{\partial \nu} v \; d\sigma - \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu} F_n(x-y) \right) v \; d\sigma, \qquad (4.20) \\ 0 &= \int_\Omega F_n(x^*-y) \;\; (-\Delta v(y)) \;\; dy + \\ &\quad + \int_{\partial\Omega} F_n(x^*-y) \;\; \frac{\partial}{\partial \nu} v \; d\sigma - \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu} F_n(x^*-y) \right) v \; d\sigma. \;\; (4.21) \end{aligned}$$

On the boundary, that is $x_1 = 0$, *it holds that* $F_n(x-y) = F_n(x^*-y)$ *and*

$$\frac{\partial}{\partial \nu} F_n(x-y) = -\frac{\partial}{\partial x_1} F_n(x-y) = \frac{\partial}{\partial x_1} F_n(x^*-y) = -\frac{\partial}{\partial \nu} F_n(x^*-y).$$

So by subtracting (4.20)-(4.21) and setting

$$G(x,y) = F_n(x-y) - F_n(x^*-y)$$

we find

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} G(x,y)\,(-\Delta v(y)) \;\; dy + \\ &\quad - \int_{\{0\} \times \mathbb{R}^{n-1}} \left(\frac{\partial}{\partial \nu} G(x,y) \right) v(0, y_2, \ldots, y_n) \;\; dy_2 \ldots dy_n. \end{aligned}$$

Since both $-\Delta v$ *and* $v_{|\{0\} \times \mathbb{R}^{n-1}}$ *are given we might have some hope to have found a solution, namely by*

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} G(x,y)\, f(y) \;\; dy + \\ &\quad - \int_{\{0\} \times \mathbb{R}^{n-1}} \left(\frac{\partial}{\partial \nu} G(x,y) \right) \varphi(y_2, \ldots y_n) \;\; dy_2 \ldots dy_n. \end{aligned}$$

Indeed this will be a solution but this conclusion does need some proof.

Definition 4.5.4 *Let* $\Omega \subset \mathbb{R}^n$ *be a domain. A function* $G : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ *such that*

$$v(x) = \int_\Omega G(x,y)\, f(y) \;\; dy - \int_{\partial\Omega} \left(\frac{\partial}{\partial \nu} G(x,y) \right) \varphi(y) \;\; d\sigma$$

is a solution of

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = \varphi & \text{on } \partial\Omega, \end{cases} \qquad (4.22)$$

is called a Green function for Ω.

Exercise 73 *Show that the* v *in the previous example is indeed a solution.*

Exercise 74 *Let* $f \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{n-1})$ *and* $\varphi_a, \varphi_b \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-2})$. *Find a solution for*

$$\begin{cases} -\Delta v(x) = f(x) & \text{for } x \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{n-2}, \\ v(0, x_2, x_3, \dots, x_n) = \varphi_a(x_2, x_3, \dots, x_n) & \text{for } x_2 \in \mathbb{R}^+, x_k \in \mathbb{R} \text{ and } k \geq 3, \\ v(x_1, 0, x_3, \dots, x_n) = \varphi_b(x_1, x_3, \dots, x_n) & \text{for } x_1 \in \mathbb{R}^+, x_k \in \mathbb{R} \text{ and } k \geq 3. \end{cases}$$

Exercise 75 *Let* $n \in \{2, 3, \dots\}$ *and define*

$$\tilde{F}_n(x,y) = \begin{cases} F_n\left(x\,|y| - \frac{y}{|y|}\right) & \text{for } y \neq 0, \\ F_n(1, 0, \dots, 0) & \text{for } y = 0. \end{cases}$$

1. *Show that* $\tilde{F}_n(x,y) = \tilde{F}_n(y,x)$ *for all* $x, y \in \mathbb{R}$ *with* $y \neq |x|^{-2} x$.

2. *Show that* $y \mapsto \tilde{F}_n(x,y)$ *is harmonic on* $\mathbb{R}^n \backslash \left\{ |x|^{-2} x \right\}$.

3. *Set* $G(x,y) = F_n(|x-y|) - \tilde{F}_n(x,y)$ *and* $B = \{x \in \mathbb{R}^n; |x| < 1\}$. *Show that*
$$v(x) = \int_B G(x,y)\, f(y)\, dy - \int_{\partial B} \left(\frac{\partial}{\partial \nu} G(x,y) \right) \varphi(y)\, d\sigma$$
is a solution of
$$\begin{cases} -\Delta v = f & \text{in } B, \\ v = \varphi & \text{on } \partial B. \end{cases} \tag{4.23}$$
You may assume that $f \in C^\infty(\bar{B})$ *and* $\varphi \in C^\infty(\partial B)$.

Exercise 76 *Consider (4.23) for* $\varphi = 0$ *and let* G *be the Green function of the previous exercise.*

1. *Let* $p > \frac{1}{2} n$. *Show that* $\mathcal{G} : L^p(B) \to L^\infty(B)$ *defined by*
$$(\mathcal{G} f)(x) := \int_B G(x,y)\, f(y)\, dy \tag{4.24}$$
is a bounded operator.

2. *Let* $p > n$. *Show that* $\mathcal{G} : L^p(B) \to W^{1,\infty}(B)$ *defined as in (4.24) is a bounded operator.*

3. *Try to improve one of the above estimates.*

4. *Compare these results with the results of section 2.5*

Exercise 77 *Suppose that* f *is real analytic and consider*

$$\begin{cases} -\Delta v = f & \text{in } B, \\ v = 0 & \text{on } \partial B, \\ \frac{\partial}{\partial \nu} v = 0 & \text{on } \partial B. \end{cases} \tag{4.25}$$

1. *What may we conclude from Cauchy-Kowalevski?*

2. *Suppose that* $f = 1$. *Compute the 'Cauchy-Kowalevski'-solution.*

Week 5

Some classics for a unique solution

5.1 Energy methods

Suppose that we are considering the heat equation or wave equation with the space variable x in a bounded domain Ω, that is

$$\begin{cases} \text{p.d.e.:} & u_t = \Delta u & \text{in } \Omega \times \mathbb{R}^+, \\ \text{initial c.:} & u(x,0) = \phi_0(x) & \text{for } x \in \Omega, \\ \text{boundary c.:} & u(x,t) = g(x,t) & \text{for } x \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{5.1}$$

or

$$\begin{cases} \text{p.d.e.:} & u_{tt} = c^2 \Delta u & \text{in } \Omega \times \mathbb{R}^+, \\ \text{initial c.:} & \begin{array}{l} u(x,0) = \phi_0(x) \\ u_t(x,0) = \phi_1(x) \end{array} & \text{for } x \in \Omega, \\ \text{boundary c.:} & u(x,t) = g(x,t) & \text{for } x \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{5.2}$$

Then it is not clear by Cauchy-Kowalevski that these problems are well-posed. Only locally near $(x,0) \in \Omega \times \mathbb{R}$ there is a unique solution for the second problem at least when the ϕ_i are real analytic. When Ω is a special domain we may construct solutions of both problems above by an appropriate orthonormal system. Even for more general domains there are ways of establishing a solution.

The question we want to address in this paragraph is whether or not such a solution in unique. For the two problems above one may proceed by considering an energy functional.

- For (5.1) this is
$$E(u)(t) = \tfrac{1}{2} \int_\Omega |u(x,t)|^2 \, dx.$$
This quantity is called the thermal energy of the body Ω at time t.

  Now let u_1 and u_2 be two solutions of (5.1) and set $u = u_1 - u_2$. This u is a solution of (5.1) with $\phi_i = 0$ and $g = 0$. Since u satisfies 0 initial and boundary conditions we obtain

$$\begin{aligned}\frac{\partial}{\partial t}E(u)(t) &= \frac{\partial}{\partial t}\left(\tfrac{1}{2}\int_\Omega |u(x,t)|^2\,dx\right) = \int_\Omega u_t(x,t)\ u(x,t)dx = \\ &= \int_\Omega \Delta u(x,t)\ u(x,t)dx = -\int_\Omega |\nabla u(x,t)|^2\,dx \le 0.\end{aligned}$$

So $t \mapsto E(u)(t)$ decreases, is nonnegative and starts with $E(u)(0) = 0$. Hence $E(u)(t) \equiv 0$ for all $t \ge 0$, which implies that $u(x,t) \equiv 0$.

Lemma 5.1.1 *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with a $\partial\Omega \in C^1$ and let $T > 0$. Then $C^2(\bar{\Omega} \times [0,T])$ solutions of (5.1) are unique.*

- For (5.2) the appropriate functional is

$$E(u)(t) = \tfrac{1}{2}\int_\Omega \left(c^2 |\nabla u(x,t)|^2 + |u_t(x,t)|^2\right) dx.$$

This quantity is called the energy of the body Ω at time t;

- $\frac{1}{2}\int_\Omega c^2 |\nabla u(x,t)|^2\,dx$ is the potential energy due to the deviation from the equilibrium,
- $\frac{1}{2}\int_\Omega |u_t(x,t)|^2\,dx$ is the kinetic energy.

Again we suppose that there are two solutions and set $u = u_1 - u_2$ such that u satisfies 0 initial and boundary conditions. Now (note that we use u is twice differentiable)

$$\begin{aligned}\frac{\partial}{\partial t}E(u)(t) &= \int_\Omega \left(c^2\nabla u(x,t)\cdot\nabla u_t(x,t) + u_t(x,t)u_{tt}(x,t)\right) dx = \\ &= \int_\Omega \left(-c^2\Delta u(x,t) + u_{tt}(x,t)\right) u_t(x,t)dx = 0.\end{aligned}$$

As before we may conclude that $E(u)(t) \equiv 0$ for all $t \ge 0$. This implies that $|\nabla u(x,t)| = 0$ for all $x \in \Omega$ and $t \ge 0$. Since $u(x,t) = 0$ for $x \in \partial\Omega$ we find that $u(x,t) = 0$. Indeed, let $x^* \in \partial\Omega$ the point closest to x, and we find

$$u(x,t) = u(x^*,t) + \int_0^1 \nabla u(\theta x + (1-\theta)x^*, t)\cdot(x - x^*)\,d\theta = 0.$$

Lemma 5.1.2 *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with a $\partial\Omega \in C^1$ and let $T > 0$. Then $C^2(\bar{\Omega} \times [0,T])$ solutions of (5.2) are unique.*

- In a closely related way one can even prove uniqueness for

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Suppose that there are two solutions and call the difference u. This u satisfies (5.3) with f and g both equal to 0. So

$$\int_\Omega |\nabla u|^2\,dx = -\int_0 \Delta u\ u\ dx = 0$$

and $\nabla u = 0$ implying, since $u_{|\partial\Omega} = 0$, that $u = 0$.

Lemma 5.1.3 *Suppose Ω is a bounded domain in $\mathbb{R}^n$ with a $\partial\Omega \in C^1$. Then $C^2(\bar{\Omega})$ solutions of (5.3) are unique.*

Exercise 78 *Why is the condition 'Ω is bounded' appearing in the three lemma's above?*

Exercise 79 *Can we prove a similar uniqueness result for the following b.v.p.?*

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = \phi_0(x) & \text{for } x \in \Omega, \\ \frac{\partial}{\partial n} u(x,t) = g(x) & \text{for } x \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{5.4}$$

Exercise 80 *And for this one?*

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = \phi_0(x) & \\ u_t(x,0) = \phi_1(x) & \text{for } x \in \Omega, \\ \frac{\partial}{\partial n} u(x,t) = g(x) & \text{for } x \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{5.5}$$

Exercise 81 *Let $\Omega = \{(x_1,x_2); -1 < x_1, x_2 < 1\}$ and let*

$$\begin{array}{c} \Gamma_\ell = \{(-1,s)\,; -1 < s < 1\}\,, \\ \Gamma_r = \{(1,s)\,; -1 < s < 1\}\,. \end{array}$$

Let $f \in C^\infty\left(\bar{\Omega}\right)$. Which of the following problems has at most one solution in $C^2\left(\bar{\Omega}\right)$? And which one has at least one solution in $W^{2,2}(\Omega)$?

$$A : \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad B : \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega. \end{cases}$$

$$C : \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_\ell, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega\backslash\Gamma_\ell. \end{cases} \qquad D : \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega, \\ u(1,1) = 0. & \end{cases}$$

$$E : \begin{cases} -\Delta u = f & \text{in } \Omega\backslash\{0\}, \\ u = 0 & \text{on } \partial\Omega, \\ u(0,0) = 0. & \end{cases} \qquad F : \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Gamma_\ell, \\ \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega\backslash\Gamma_r. \end{cases}$$

Exercise 82 *For which of the problems in the previous exercise can you find $f \in C^\infty(\bar{\Omega})$ such that no solution in $C^2\left(\bar{\Omega}\right)$ exists?*

1. *Can you find a condition on λ such that for the solution u of*

$$\begin{cases} u_t = \Delta u + \lambda u & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = \phi_0(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{5.6}$$

   the norm $\|u(\cdot,t)\|_{L^2(\Omega)}$ remains bounded for $t \to \infty$.

2. *Can you find a condition on f such that*

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,0) = \phi_0(x) & \text{for } x \in \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{5.7}$$

   $\|u(\cdot,t)\|_{L^2(\Omega)}$ remains bounded for $t \to \infty$.

Exercise 83 *Here is a model for a string with some damping:*

$$\begin{cases} u_{tt} + a u_t = c^2 u_{xx} & \text{in } (0,\ell)\times\mathbb{R}^+, \\ u(x,0) = \phi(x) & \text{for } 0<x<\ell, \\ u_t(x,0) = 0 & \text{for } 0<x<\ell, \\ u(x,t) = 0 & \text{for } x\in\{0,\ell\} \text{ and } t\geq 0. \end{cases} \tag{5.8}$$

Find a bound like $\left\|u_x^2(\cdot,t)\right\|_{L^2} \leq M e^{-bt}$.

Exercise 84 *A more realistic model is*

$$\begin{cases} u_{tt} + a\,|u_t|\,u_t = c^2 u_{xx} & \text{in } (0,\ell)\times\mathbb{R}^+, \\ u(x,0) = \phi(x) & \text{for } 0<x<\ell, \\ u_t(x,0) = 0 & \text{for } 0<x<\ell, \\ u(x,t) = 0 & \text{for } x\in\{0,\ell\} \text{ and } t\geq 0. \end{cases} \tag{5.9}$$

Can you show that $u(\cdot,t)\to 0$ *for* $t\to\infty$ *in some norm?*

5.2 Maximum principles

In this section we will consider second order linear partial differential equations in $\mathbb{R}^n$. The corresponding differential operators are as follows:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \tag{5.10}$$

with $a_{ij}, b_i, c \in C(\bar{\Omega})$. Without loss of generality we may assume that $a_{ij} = a_{ji}$.

In order to restrict ourselves to parabolic and elliptic cases with the right sign for stating the results below we will assume a sign for the operator.

Definition 5.2.1 *We will call L parabolic-elliptic in x if*

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq 0 \text{ for all } \xi \in \mathbb{R}^n. \tag{5.11}$$

We will call L elliptic in x if there is $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geq \lambda \left|\xi\right|^2 \text{ for all } \xi \in \mathbb{R}^n. \tag{5.12}$$

Definition 5.2.2 *For $\Omega \subset \mathbb{R}^n$ and L as in (5.10)-(5.11) we will define the parabolic boundary $\partial_L \Omega$ as follows:*
$x \in \partial_L \Omega$ if $x \in \partial\Omega$ and either

1. *$\partial\Omega$ is not C^1 locally near x, or*
2. *$\sum_{i,j=1}^{n} a_{ij}(x)\, \nu_i \nu_j > 0$ for the outside normal ν at x, or*
3. *$\sum_{i,j=1}^{n} a_{ij}(x)\, \nu_i \nu_j = 0$ and $\sum_{i=1}^{n} b_i(x) \nu_i \geq 0$ for the outside normal ν at x.*

Lemma 5.2.3 *Let L be parabolic-elliptic on $\bar{\Omega}$ with $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \partial\Omega \backslash \partial_L \Omega$ with $\Gamma \in C^1$. Suppose that $u \in C^2(\Omega \cup \Gamma)$, that $Lu > 0$ in $\Omega \cup \Gamma$ and that $c \leq 0$ on $\Omega \cup \Gamma$. Then u cannot attain a nonnegative maximum in $\Omega \cup \Gamma$.*

Example 5.2.4 *For the heat-equation with source term $u_t - \Delta u = f$ on $\Omega = \tilde{\Omega} \times (0, T) \subset \mathbb{R}^{n+1}$ the non-parabolic boundary is $\tilde{\Omega} \times \{T\}$. Indeed, rewriting to $\Delta u - u_t$ we find part 2 of Definition 5.2.2 satisfied on $\partial\Omega \times (0, T)$ and part 3 of Definition 5.2.2 satisfied on $\tilde{\Omega} \times \{0\}$. In (x, t)-coordinates $b = (0, \ldots, 0, -1)$ one finds that $a_{ij} = \delta_{ij}$ for all $i, j \neq (n + 1, n + 1)$ and $a_{n+1,n+1} = 0$. So*

$$\sum_{i,j=1}^{n+1} a_{ij}(x)\, \nu_i \nu_j = \sum_{i,j=1}^{n} \delta_{ij} 0^2 + 0\ 1^2 = 0 \text{ and } b \cdot \nu = -1 \text{ on } \Omega \times \{T\}.$$

So, if $u_t - \Delta u > 0$ then u cannot attain a nonpositive minimum in $\Omega \times (0, T]$ for every $T > 0$. In other words, if u satisfies

$$\begin{cases} u_t - \Delta u > 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u(x, 0) = \phi_0(x) \geq 0 & \text{for } x \in \Omega, \\ u(x, t) = g(x, t) \geq 0 & \text{for } x \in \partial\Omega \times \mathbb{R}. \end{cases}$$

then $u \geq 0$ in $\bar{\Omega} \times \mathbb{R}_0^+$.

Proof. Suppose that u does attain a nonnegative maximum in $\Omega \cup \Gamma$, say in x_0. If $x_0 \in \Omega$ then $\nabla u(x_0) = 0$. If $x_0 \in \Gamma$ then by assumption $b \cdot \nu \leq 0$ and $\nabla u(x_0) = \gamma \nu$ for some nonnegative constant γ. So in both case

$$b(x_0) \cdot \nabla u(x_0) + c(x_0) u(x_0) \leq 0.$$

We find that

$$\sum_{i,j=1}^{n} a_{ij}(x_0) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x_0) \geq Lu(x_0) > 0.$$

As before we may diagonalize $(a_{ij}(x_0)) = T^t D T$ and find in the new coordinates $y = Tx$ with $U(y) = u(x)$ that

$$\sum_{i=1}^{n} d_{ii}^2 \frac{\partial^2}{\partial y_i^2} U(Tx_0) > 0$$

with $d_{ii} \geq 0$ by our assumption in (5.11).

If $x_0 \in \Omega$ then in a maximum $\frac{\partial^2}{\partial y_i^2} U(Tx_0) \leq 0$ for all $i \in \{1, n\}$ and we find

$$\sum_{i=1}^{n} d_{ii}^2 \frac{\partial^2}{\partial y_i^2} U(Tx_0) \leq 0, \tag{5.13}$$

a contradiction.

If $x_0 \in \Gamma$ then ν is an eigenvector of $(a_{ij}(x_0))$ with eigenvalue zero and hence we may use this as our first new basis element and one for which we have $d_{11} = 0$. In a maximum we still have $\frac{\partial^2}{\partial y_i^2} U(Tx_0) \leq 0$ for all $i \in \{2, n\}$. Our proof is saved since the first component is killed through $d_{11} = 0$ and we still find the contradiction by (5.13). ■

In the remainder we will restrict ourselves to the strictly elliptic case.

Definition 5.2.5 *The operator L as in (5.10) is called strictly elliptic on Ω if there is $\lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \tag{5.14}$$

Exercise 85 *Suppose that $\Omega \subset \mathbb{R}^n$ is bounded. Show that if L as in (5.10) is elliptic for all $x \in \bar{\Omega}$ then L is strictly elliptic on Ω as in (5.14). Also give an example of an elliptic L that is not strictly elliptic.*

For $u \in C(\bar{\Omega})$ we define $u^+ \in C(\bar{\Omega})$ by

$$u^+(x) = \max[0, u(x)] \text{ and } u^-(x) = \max[0, -u(x)].$$

Theorem 5.2.6 (Weak Maximum Principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and suppose that L is strictly elliptic on Ω with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Lu \geq 0$ in Ω, then the maximum of u^+ is attained at the boundary.*

Proof. The proof of this theorem relies on a smartly chosen auxiliary function. Supposing that $\Omega \subset B_R(0)$ we consider w defined by

$$w(x) = u(x) + \varepsilon e^{\alpha x_1}$$

with $\varepsilon > 0$. One finds that

$$Lw(x) = Lu(x) + \varepsilon \left(\alpha^2 a_{11}(x) + \alpha b_1(x) + c(x)\right) e^{\alpha x_1}.$$

Since $a_{11}(x) \geq \lambda$ be the strict ellipticity assumption and since $b_1, c \in C(\bar{\Omega})$ with Ω bounded, we may choose α such that

$$\alpha^2 a_{11}(x) + \alpha b_1(x) + c(x) e^{\alpha x_1} > 0.$$

The result follows for w from the previous Lemma and hence

$$\sup_{\Omega} u \leq \sup_{\Omega} w \leq \sup_{\Omega} w^+ \leq \sup_{\partial\Omega} w^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon e^{\alpha R}.$$

Letting $\varepsilon \downarrow 0$ the claim follows. ■

Exercise 86 *State and prove a weak maximum principle for* $L = -\frac{\partial}{\partial t} + \Delta$ *on* $\Omega \times (0, T)$.

Theorem 5.2.7 (Strong Maximum Principle) *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and suppose that* L *is strictly elliptic on* Ω *with* $c \leq 0$. *If* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *and* $Lu \geq 0$ *in* Ω, *then either*

1. $u \equiv \sup\left\{u(x); x \in \bar{\Omega}\right\}$, *or*

2. u *does not attain a nonnegative maximum in* Ω.

For the formulation of a boundary version of the strong maximum principle we need a condition on Ω.

Definition 5.2.8 *A domain* $\Omega \subset \mathbb{R}^n$ *satisfies the interior sphere condition at* $x_0 \in \partial\Omega$ *if there is a open ball* B *such that* $B \subset \Omega$ *and* $x_0 \in \partial B$.

Theorem 5.2.9 (Hopf's boundary point lemma) *Let* $\Omega \subset \mathbb{R}^n$ *be a domain that satisfies the interior sphere condition at* $x_0 \in \partial\Omega$. *Let* L *be strictly elliptic on* Ω *with* $c \leq 0$. *If* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfies* $Lu \geq 0$ *in* Ω *and is such that* $\max\left\{u(x); x \in \bar{\Omega}\right\} = u(x_0) \geq 0$, *then either*

1. $u \equiv u(x_0)$ *on* $\bar{\Omega}$, *or*

2. $\liminf\limits_{t \downarrow 0} \dfrac{u(x_0) - u(x_0 + t\mu)}{t} > 0$ *(possibly* $+\infty$*) for every direction* μ *pointing into an interior sphere.*

Remark 5.2.10 *If* $u \in C^1(\Omega \cup \{x_0\})$ *then*

$$\frac{\partial}{\partial \mu} u(x_0) = -\liminf_{t \downarrow 0} \frac{u(x_0) - u(x_0 + t\mu)}{t} < 0,$$

which means for the outward normal ν

$$\frac{\partial}{\partial \nu} u(x_0) > 0$$

Exercise 87 *Suppose that* $\Omega = B_r(x_a) \cup B_r(x_b)$ *with* $r = 2|x_a - x_b|$. *Let* L *be strictly elliptic on* Ω *with* $c \leq 0$ *and assume that* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *satisfies*

$$\begin{cases} Lu \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Show that if $u \in C^1(\bar{\Omega})$ *then* $u \equiv 0$ *in* $\bar{\Omega}$. *Hint: consider some* $x_0 \in \partial B_r(x_a) \cap \partial B_r(x_b)$.

Exercise 88 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* L *be strictly elliptic on* Ω *(and we put no sign restriction on* c*). Suppose that there is a function* $v \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* $-Lv \geq 0$ *in* Ω *and* $v > 0$ *on* $\bar{\Omega}$.
Show that any function $w \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that*

$$\begin{cases} -Lw \geq 0 & \text{in } \Omega, \\ w \geq 0 & \text{on } \partial\Omega, \end{cases}$$

is nonnegative.

Exercise 89 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* L *be strictly elliptic on* Ω *with* $c \leq 0$. *Let* f *and* ϕ *be given functions. Show that*

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases}$$

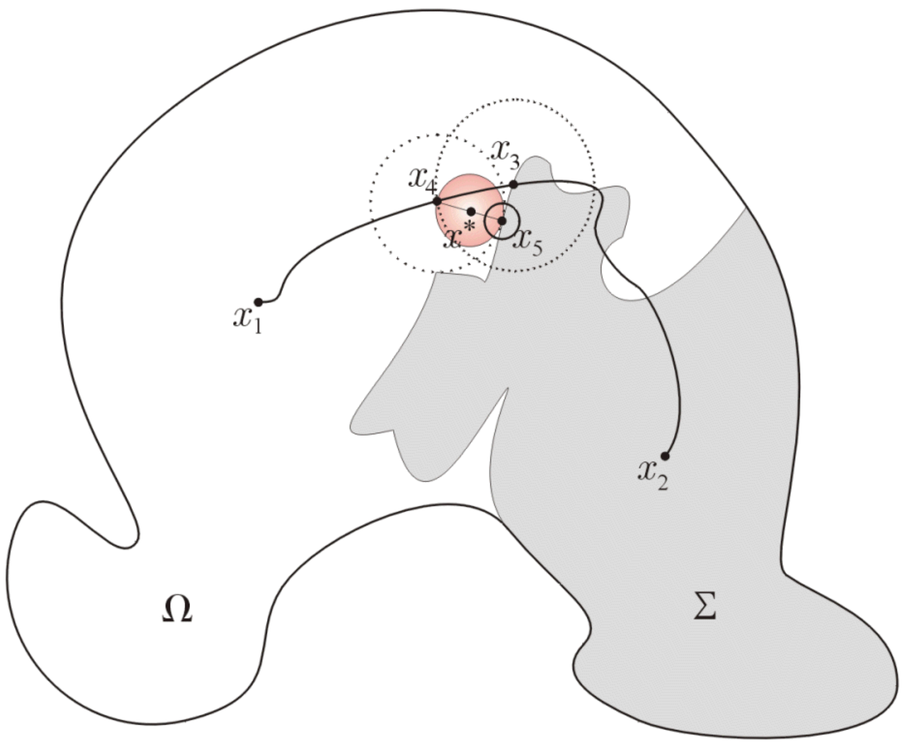
has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$.

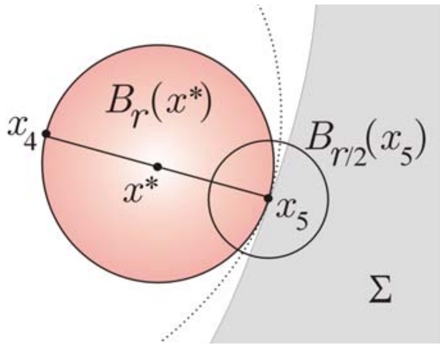
5.3 Proof of the strong maximum principle

Set $m = \sup_\Omega u$ and set $\Sigma = \{x \in \Omega; u(x) = m\}$. We have to prove that when $m > 0$ then either Σ is empty or that $\Sigma = \Omega$. We argue by contradiction and assume that both Σ and $\Omega \backslash \Sigma$ are nonempty, say $\Omega \backslash \Sigma \ni x_1$ and $\Sigma \ni x_2$.

I. Constructing an appropriate subdomain. First we will construct a ball in Ω that touches Σ exactly once. Since u is continuous $\Omega \backslash \Sigma$ is open. Move along the arc from x_1 to x_2 and there will be a first x_3 on this arc that lies in Σ. Set s the minimum of $|x_1 - x_3|$ and the distance of x_3 to $\partial\Omega$. Take x_4 on the arc between x_1 and x_3 such that $|x_3 - x_4| < \frac{1}{2}s$.

Next we take $B_{r_1}(x_4)$ to be the largest ball around x_4 that is contained in $\Omega \backslash \Sigma$. Since $|x_3 - x_4| < \frac{1}{2}s$ we find that $r_1 \leq \frac{1}{2}s$ and that there is at least one point of Σ on $\partial B_{r_1}(x_4)$. Let us call such a point x_5. The final step of the construction is to set $x^* = \frac{1}{2}x_4 + \frac{1}{2}x_5$ and $r = \frac{1}{2}r_1$. The ball $B_r(x^*)$ is contained in $\Omega \backslash \Sigma$ and $\partial B_r(x^*) \cap \Sigma = \{x_5\}$. We will also use the ball $B_{\frac{1}{2}r}(x_5)$. See the pictures below.





II. The auxiliary function. Next we set for $\alpha > 0$ the auxiliary function

$$h(x) = \frac{e^{-\frac{\alpha}{2}|x-x^*|^2} - e^{-\frac{\alpha}{2}r^2}}{1 - e^{-\frac{\alpha}{2}r^2}}.$$

One should notice that this function is tailored in such a way that $h = 0$ for $x \in \partial B_r(x^*)$, positive inside $B_r(x^*)$ and negative outside of $\overline{B_r(x^*)}$. Moreover $h(x) \leq 1$ on $B_r(x^*)$.

On $B_{\frac{1}{2}r}(x_5)$ one finds, using that $|x - x^*| \in \left(\frac{1}{2}r, \frac{3}{2}r\right)$ that

$$Lh = \frac{\alpha^2 \sum\limits_{i,j=1}^{n} a_{ij}(x_i - x_i^*)(x_j - x_j^*) - \alpha \sum\limits_{i=1}^{n} (a_{ii} + b_i(x_i - x_i^*) + c}{1 - e^{-\frac{\alpha}{2}r^2}} e^{-\frac{\alpha}{2}|x-x^*|^2} +$$

$$-c(x)\frac{e^{-\frac{\alpha}{2}r^2}}{1 - e^{-\frac{\alpha}{2}r^2}} \geq$$

$$\geq \left(4\alpha^2\lambda \left(\tfrac{1}{2}r\right)^2 - 2\alpha \sum_{i=1}^{n} \left(a_{ii}(x) + |b_i(x)| \tfrac{3}{2}r\right) + c\right) \frac{e^{-\frac{\alpha}{2}|x-x^*|^2}}{1 - e^{-\frac{\alpha}{2}r^2}}$$

and by choosing α large enough we find $Lh > 0$ in $B_{\frac{1}{2}r}(x_5)$.

III. Deriving a contradiction. As before we consider $w = u + \varepsilon h$. Now the subdomain that we will use to derive a contradiction is $B_{\frac{1}{2}r}(x_5)$. The boundary of this set consists of two parts:

$$\Gamma_1 = \partial B_{\frac{1}{2}r}(x_5) \cap \overline{B_r(x^*)} \text{ and } \Gamma_2 = \partial B_{\frac{1}{2}r}(x_5) \backslash \Gamma_1.$$

Since Γ_1 is closed, since $u_{|\Gamma_1} < m$ by construction and since u is continuous we find

$$\sup \{u(x); x \in \Gamma_1\} = \max \{u(x); x \in \Gamma_1\} = \tilde{m} < m.$$

On Γ_2 we have $h \leq 0$ and hence $w = u + \varepsilon h \leq u \leq m$.

Next we choose $\varepsilon > 0$ such that

$$\varepsilon = \frac{1}{2}(m - \tilde{m})$$

As a consequence we find that

$$\sup \left\{ w(x); x \in \partial B_{\frac{1}{2}r}(x_5) \right\} < m.$$

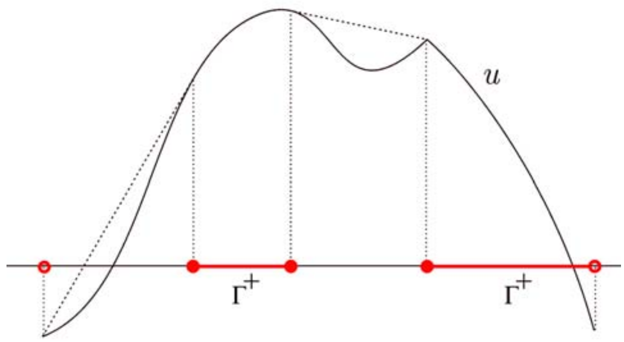
Since $w(x_5) = u(x_5) = m$ and also $Lw > 0$ in $B_{\frac{1}{2}r}(x_5)$ we obtain a contradiction with Theorem 5.2.6.

5.4 Alexandrov's maximum principle

Alexandrov's maximum principle gives a first step towards regularity. Before we are able to state the result we need a definition.

Definition 5.4.1 *For $u \in C(\bar{\Omega})$ with Ω a domain in $\mathbb{R}^n$, the upper contact set Γ^+ is defined by*

$$\Gamma^+ = \{y \in \Omega; \exists a_y \in \mathbb{R}^n \text{ such that } \forall x \in \Omega : u(x) \leq u(y) + a_y \cdot (x-y)\}.$$



Next we define a lemma concerning this upper contact set. Here the diameter of Ω will play a role:

$$\mathrm{diam}(\Omega) = \sup\{|x-y|; x, y \in \Omega\}.$$

Lemma 5.4.2 *Suppose that Ω is bounded. Let $g \in C(\mathbb{R}^n)$ be nonnegative and $u \in C(\bar{\Omega}) \cap C^2(\Omega)$. Set*

$$M = (\mathrm{diam}(\Omega))^{-1} \left(\sup_{\Omega} u - \sup_{\partial\Omega} u \right). \tag{5.15}$$

Then

$$\int_{B_M(0) \subset \mathbb{R}^n} g(z)dz \leq \int_{\Gamma^+} g(\nabla u(x)) \left| \det \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \right) \right| dx. \tag{5.16}$$

Proof. Let $\Sigma \subset \mathbb{R}^n$ be the following set:

$$\Sigma = \left\{\nabla u(x); x \in \Gamma^+\right\}.$$

If $\nabla u : \Gamma^+ \to \Sigma$ is a bijection then

$$\int_{\Sigma} g(z)dz = \int_{\Gamma^+} g(\nabla u(x)) \left| \det \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \right) \right| dx.$$

is just a consequence of a change of variables. One cannot expect that this holds but since the mapping is onto and since $g \geq 0$ one finds

$$\int_{\Sigma} g(z)dz \leq \int_{\Gamma^+} g(\nabla u(x)) \left| \det \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \right) \right| dx.$$

So if we can show that $B_M(0) \subset \Sigma$ we are done. In other words, it is sufficient to show that for every $a \in B_M(0)$ there is $y \in \Gamma^+$ such that $a = \nabla u(y)$.

Set

$$\ell(a) = \min_{x \in \bar{\Omega}} (a \cdot x - u(x))$$

which is attained at some y_a since $\bar{\Omega}$ is closed and bounded and u is continuous. So we have

$$\begin{array}{rcl} a \cdot y_a - u(y_a) & = & \ell(a) \text{ for some } y_a \in \bar{\Omega}, \\ a \cdot x - u(x) & \geq & \ell(a) \text{ for all } x \in \bar{\Omega}, \end{array}$$

which implies that

$$u(y_a) \geq u(x) + a \cdot (y_a - x) \text{ for all } x \in \bar{\Omega}. \tag{5.17}$$

By taking $x_0 \in \bar{\Omega}$ such that $u(x_0) = \max_{x \in \bar{\Omega}} u(x)$ one obtains

$$\begin{array}{rcl} u(y_a) & \geq & u(x_0) + a \cdot (y_a - x_0) \\ & = & \sup\limits_{x \in \Omega} u + a \cdot (y_a - x_0) \\ & = & \sup\limits_{x \in \partial\Omega} u + M \operatorname{diam}(\Omega) + a \cdot (y_a - x_0) \\ & > & \sup\limits_{x \in \partial\Omega} u + M \operatorname{diam}(\Omega) - M |y_a - x_0| \geq \sup\limits_{x \in \partial\Omega} u. \end{array}$$

So $y_a \notin \partial\Omega$ and hence $y_a \in \Omega$. Since (5.17) holds and since u is differentiable in y_a we find that $a = \nabla u(y_a)$. By this construction we even have that $y_a \in \Gamma^+$, the upper contact set. ■

Corollary 5.4.3 *Suppose that Ω is bounded and that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$. Denote the volume of the unit ball in $\mathbb{R}^n$ by ω_n^* and set*

$$\mathcal{D}^*(x) = \left(\det \left(a_{ij}(x) \right) \right)^{1/n}.$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{\sqrt[n]{\omega_n^*}} \left\| \frac{-\sum_{i.j=1}^n a_{ij}(\cdot) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(\cdot)}{n \mathcal{D}^*(\cdot)} \right\|_{L^n(\Gamma^+)}.$$

Remark 5.4.4 *Replacing Ω with $\Omega^+ = \{x \in \Omega; u(x) > 0\}$ one finds*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{\operatorname{diam}(\Omega)}{\sqrt[n]{\omega_n^*}} \left\| \frac{-\sum_{i.j=1}^n a_{ij}(\cdot) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(\cdot)}{n \mathcal{D}^*(\cdot)} \right\|_{L^n(\Gamma^+ \cap \Omega^+)}.$$

Proof. Note that

$$\sum_{i.j=1}^n a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) = \operatorname{trace}(\mathbf{AD})$$

with the matrices $\mathbf{A}$ and $\mathbf{D}$ defined by

$$\mathbf{A} = \left(a_{ij}(x) \right) \text{ and } \mathbf{D} = \left(\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} u(x) \right).$$

Now let us diagonalize the elliptic operator $\sum_{i.j=1}^{n} a_{ij}(\bar{x})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$ by choosing new orthonormal coordinates y at this point $\bar{x}$ (the new coordinates will be different at each point $\bar{x}$ but that doesn't matter since for the moment we are just using linear algebra at this fixed point $\bar{x}$) which fit the eigenvalues of $(a_{ij}(x))$. So $\mathbf{AD} = T^t \mathbf{\Lambda}\tilde{\mathbf{D}} T$ with diagonal matrices

$$\mathbf{\Lambda} = \Big(\lambda_i(\bar{x}) \Big) \text{ and } \tilde{\mathbf{D}} = \left(\frac{\partial^2}{\partial y_i^2} (v \circ T)(y) \right).$$

Since for T orthonormal the eigenvalues of $\mathbf{AD}$ and $\Lambda\tilde{\mathbf{D}}$ coincide and

$$\begin{aligned} \operatorname{trace}(\mathbf{AD}) &= \operatorname{trace}\left(\Lambda\tilde{\mathbf{D}}\right) \\ \det(\mathbf{AD}) &= \det\left(\Lambda\tilde{\mathbf{D}}\right) \end{aligned}$$

we may consider the at $\bar{x}$ diagonalized elliptic operator. The ellipticity condition implies $\lambda_i(\bar{x}) \geq \lambda > 0$. Since we are at a point in the upper contact set Γ^+ we have $\frac{\partial^2}{\partial y_i^2} u(y) \leq 0$. So on Γ^+ the matrix $T^t \mathbf{\Lambda}\tilde{\mathbf{D}} T$ is nonpositive definite. Writing $\mu_i = -\lambda_i(\bar{x})\frac{\partial^2}{\partial y_i^2}(v \circ T)(y)$ for the the eigenvalues of $-\mathbf{AD}$ by μ_i and using that the geometric mean is less then the arithmetic mean (for positive numbers) we obtain

$$\begin{aligned} \mathcal{D}^*(x) \det(-\mathbf{D})^{1/n} &= \det(-\mathbf{AD})^{1/n} = \left(\prod_{i=1}^{n} \mu_i \right)^{1/n} \leq \\ &\leq \frac{1}{n} \sum_{i=1}^{n} \mu_i = \frac{1}{n} \operatorname{trace}(-\mathbf{AD}). \end{aligned}$$

So we may conclude that for every $x \in \Gamma^+$

$$\det\left(-\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u(x) \right) \leq \left(\frac{-\sum_{i.j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u(x)}{n\ \mathcal{D}^*(x)} \right)^n. \tag{5.18}$$

Taking $g = 1$ in (5.16) and M as in (5.15) we have

$$\begin{aligned} \omega_n^* \left(\frac{\sup_\Omega u - \sup_{\partial\Omega} u}{\operatorname{diam}(\Omega)} \right)^n &= \int_{B_M(0)} 1 dz \leq \\ &\leq \int_{\Gamma^+} \left| \det\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u(x) \right) \right| dx \leq \\ &\leq \int_{\Gamma^+} \left(\frac{-\sum_{i.j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u(x)}{n\ \mathcal{D}^*(x)} \right)^n dx, \end{aligned}$$

which completes the proof. ■

Theorem 5.4.5 (Alexandrov's Maximum Principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let L be elliptic as in (5.12) with $c \leq 0$. Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq f$ with*

$$\frac{|b(\cdot)|}{\mathcal{D}^*(\cdot)}, \frac{f(\cdot)}{\mathcal{D}^*(\cdot)} \in L^n(\Omega),$$

and let Γ^+ *denote the upper contact set of* u. *Then one finds*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C \ \operatorname{diam}(\Omega) \ \left\| \frac{f^-(\cdot)}{\mathcal{D}^*(\cdot)} \right\|_{L^n(\Gamma^+)},$$

where C *depends on* n *and* $\left\| \frac{|b(\cdot)|}{\mathcal{D}^*(\cdot)} \right\|_{L^n(\Gamma^+)}$.

Proof. If the b_i and c components of L would be equal 0 then the result follows from (5.4.3). Indeed, on Γ^+

$$0 \le -\sum_{i.j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \le -f(x) \le f^-(x).$$

For nonzero b_i and c we proceed as follows. Since it is sufficient to consider $u > 0$ we may restrict ourselves to Ω^+. Then we have $c(x)u(x) \le 0$ and hence for any $\mu > 0$:

$$\begin{aligned}
0 \ &\le \ -\sum_{i.j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \le \\
&\le \ \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x)u(x) - f(x) \le \\
&\le \ \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x) + f^-(x) \le \text{(by Cauchy-Schwarz)} \\
&\le \ |b_i(x)| \ \ |\nabla u(x)| \ \ 1 + \mu^{-1} f^-(x) \ \mu \ 1 \le \text{(by Hölder)} \\
&\le \ \left(|b_i(x)|^n + \left|\mu^{-1} f^-(x)\right|^n \right)^{\frac{1}{n}} \left(|\nabla u(x)|^n + \mu^n \right)^{\frac{1}{n}} (1+1)^{\frac{n-2}{n}} .
\end{aligned}$$

Using Lemma 5.4.2 on Ω^+ and with $g(z) = \left(|z|^n + \mu^n\right)^{-1}$ one finds with

$$\tilde{M} = (\operatorname{diam}(\Omega))^{-1} \left(\sup_{\Omega} u - \sup_{\partial\Omega} u^+ \right)$$

that, similar as before in (5.18),

$$\begin{aligned}
\int_{B_{\tilde{M}}(0)} \frac{1}{|z|^n + \mu^n} dz &\le \int_{\Gamma^+ \cap \Omega^+} \frac{1}{|\nabla u(x)|^n + \mu^n} \left| \det \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x) \right) \right| dx \\
\le \ & \int_{\Gamma^+ \cap \Omega^+} \frac{1}{|\nabla u(x)|^n + \mu^n} \left(\frac{-\sum_{i.j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u(x)}{n \ \mathcal{D}^*(x)} \right)^n dx \le \\
\le \ & 2^{n-2} \int_{\Gamma^+ \cap \Omega^+} \left(\frac{\left(|b_i(x)|^n + \left|\mu^{-1} f^-(x)\right|^n \right)^{\frac{1}{n}}}{n \ \mathcal{D}^*(x)} \right)^n dx \le \\
\le \ & \frac{2^{n-2}}{n^n} \int_{\Gamma^+ \cap \Omega^+} \frac{|b_i(x)|^n + \left|\mu^{-1} f^-(x)\right|^n}{\mathcal{D}^*(x)^n} dx = \\
= \ & \frac{2^{n-2}}{n^n} \left(\left\| \frac{|b_i|}{\mathcal{D}^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} + \mu^{-n} \left\| \frac{f^-}{\mathcal{D}^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} \right).
\end{aligned}$$

By a direct computation

$$\int_{B_{\tilde{M}}(0)\subset\mathbb{R}^n} \frac{1}{|z|^n+\mu^n}dz = n\omega_n^* \int_0^M \frac{r^{n-1}}{r^n+\mu^n}dr = \omega_n^* \log\left(\frac{\tilde{M}^n+\mu^n}{\mu^n}\right).$$

Choosing $\mu = \|f^-/\mathcal{D}^*\|_{L^n(\Gamma^+\cap\Omega^+)}$ it follows that

$$\omega_n^* \log\left(\left(\frac{\tilde{M}}{\|f^-/\mathcal{D}^*\|_{L^n(\Gamma^+\cap\Omega^+)}}\right)^n + 1\right) \le \frac{2^{n-2}}{n^n}\left(\left\|\frac{|b_i|}{\mathcal{D}^*}\right\|^n_{L^n(\Gamma^+\cap\Omega^+)} + 1\right)$$

and hence

$$\left(\frac{\tilde{M}}{\|f^-/\mathcal{D}^*\|_{L^n(\Gamma^+\cap\Omega^+)}}\right)^n \le e^{\frac{2^{n-2}}{n^n\omega_n^*}\left(\left\|\frac{|b_i|}{\mathcal{D}^*}\right\|^n_{L^n(\Gamma^+\cap\Omega^+)}+1\right)} - 1$$

and the claim follows. ■

5.5 The wave equation

5.5.1 3 space dimensions

This is the differential equation with four variables $x \in \mathbb{R}^3$ and $t \in \mathbb{R}^+$ and the unknown function u

$$u_{tt} = c^2 \Delta u, \tag{5.19}$$

where $\Delta u = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

Believing in Cauchy-Kowalevski an appropriate initial value problem would be

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ u(x,0) = \varphi_0(x) & \text{for } x \in \mathbb{R}^3, \\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{5.20}$$

We will start with the special case that u is radially symmetric in x, y, z. Under this assumption and setting $r = \sqrt{x^2 + y^2 + z^2}$ the equation changes in

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right).$$

Next we use the substitution

$$U(r,t) = ru(r,t),$$

and with some elementary computations

$$u_{tt} = \frac{1}{r} U_{tt} \text{ and } u_{rr} + \frac{2}{r} u_r = \frac{1}{r} U_{rr},$$

we return to the one dimensional wave equation

$$U_{tt} = c^2 U_{rr}.$$

This equation on $\mathbb{R} \times \mathbb{R}_0^+$ has solutions of the form

$$U(r,t) = \Phi(r - ct) + \Psi(r + ct).$$

Returning to u it means we obtain solutions

$$u(x,t) = \frac{1}{|x|} \Phi(|x| - ct) + \frac{1}{|x|} \Psi(|x| + ct)$$

but this does not look like covering all solutions and in fact the second term is suspicious. If Ψ is somewhere nonzero, say at x_0, then $\frac{1}{|x|}\Psi(|x|+ct)$ is unbounded for $t = c^{-1}|x_0|$ in $x = 0$. So we are just left with $u(x,t) = \frac{1}{|x|}\Phi(|x| - ct)$.

How can we turn this into a solution of (5.20)? Remember that for the heat equation it was sufficient to have the solution for the δ-function in order to find a solution for arbitrary initial values by a convolution. In the present case one might try to consider as this special 'function' (really a distribution)

$$v(x,t) = \frac{1}{|x|} \delta(|x| - ct) = \frac{1}{ct} \delta(|x| - ct).$$

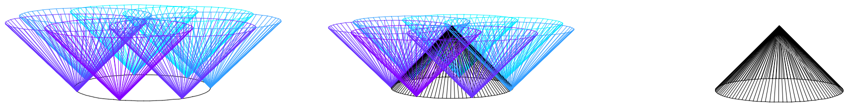
So at time t the influence of an initial 'function' at $x = 0$ distributes itself over a circle of radius ct around 0. Or similarly if we start at y, at time t the influence of an initial 'function' at $x = y$ distributes itself over a circle of radius ct around y :

$$v_1(x,t) = \frac{1}{|x-y|}\delta\left(|x-y| - ct\right) = \frac{1}{ct}\delta\left(|x-y| - ct\right).$$

Combining these 'solutions' for $y \in \mathbb{R}^n$ with density $f(y)$ we obtain

$$\begin{aligned} u\left(x,t\right) &= \int_{\mathbb{R}^3} \frac{1}{ct}\delta\left(|x-y| - ct\right) f(y)dy = \\ = \frac{1}{ct}\int_{|x-y|=ct} f(y)d\sigma_y &= \frac{1}{ct}\int_{|z|=ct} f(x+z)d\sigma_z = ct\int_{|\omega|=1} f\left(x+ct\omega\right)d\omega. \end{aligned}$$

Looking backwards from (x,t) we should see an average of the initial function at positions at distance ct from x.



On the left the cones that represent the influence of the initial data; on the right the cone representing the dependence on the initial data.

Now we have to find out what $u(x,0)$ and $u_t(x,0)$ are. For $f \in C_0^\infty(\mathbb{R}^3)$ one gets

$$u\left(x,0\right) = \lim_{t\downarrow 0} ct\int_{|\omega|=1} f\left(x+ct\omega\right)d\omega = 0$$

and

$$\begin{aligned} u_t(x,0) &= \lim_{t\downarrow 0}\frac{\partial}{\partial t}\left(ct\int_{|\omega|=1} f\left(x+ct\omega\right)d\omega\right) = \\ &= \lim_{t\downarrow 0}\left(c\int_{|\omega|=1} f\left(x+ct\omega\right)d\omega + c^2t^2\int_{|\omega|=1}\nabla f\left(x+ct\omega\right)\cdot\omega\ d\omega\right) \\ &= 4\pi c f\left(x\right). \end{aligned}$$

So there is a possible solution of (5.20) for $\varphi_0 = 0$, namely

$$\tilde{u}(\varphi_1;x,t) = \frac{1}{4\pi c^2 t}\int_{|x-y|=ct}\varphi_1\left(y\right)d\sigma. \tag{5.21}$$

Since we do have a solution with $u\left(x,0\right) = 0$ and $u_t(x,0) = \varphi_1(x)$ it could be worth a try to consider the derivative with respect to t of this $\tilde{u}$ in (5.21), with φ_1 replaced by φ_0, in order to solve the first initial condition. So $v(x,t) = \frac{\partial}{\partial t}\tilde{u}(\varphi_0;x,t)$ will satisfy the differential equation and the first initial condition, that is, $v(x,0) = \varphi_0(x)$. We still have to see about $v_t(x,0)$. First let us write out $v(x,t)$:

$$v(x,t) = \frac{\partial}{\partial t}\tilde{u}(\varphi_0; x, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|x-y|=ct}\varphi_0\left(y\right)d\sigma\right) =$$

$$\begin{aligned}
&= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{|\omega|=1}\varphi_0\left(x+ct\omega\right)d\omega\right) = \\
&= \frac{1}{4\pi}\int_{|\omega|=1}\varphi_0\left(x+ct\omega\right)d\omega + \frac{ct}{4\pi}\int_{|\omega|=1}\nabla\varphi_0\left(x+ct\omega\right)\cdot\omega\ d\omega = \\
&= \frac{1}{4\pi c^2 t^2}\int_{|x-y|=ct}\varphi_0\left(y\right)d\sigma + \frac{1}{4\pi c^2 t^2}\int_{|x-y|=ct}\nabla\varphi_0\left(y\right)\cdot\left(y-x\right)\ d\sigma,
\end{aligned}$$

We find that

$$\begin{aligned}
v_t(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi}\int_{|\omega|=1}\varphi_0\left(x+ct\omega\right)d\omega + \frac{ct}{4\pi}\int_{|\omega|=1}\nabla\varphi_0\left(x+ct\omega\right)\cdot\omega\ d\omega\right) \\
&= \frac{2c}{4\pi}\int_{|\omega|=1}\nabla\varphi_0\left(x+ct\omega\right)\cdot\omega\ d\omega + \\
&\quad + \frac{c^2 t}{4\pi}\int_{|\omega|=1}\sum_{i,j=1}^{3}\omega_i\omega_j\frac{\partial^2}{\partial x_i\partial x_j}\varphi_0\left(x+ct\omega\right)\ d\omega
\end{aligned}$$

and since

$$\lim_{t\downarrow 0}\frac{2c}{4\pi}\int_{|\omega|=1}\nabla\varphi_0\left(x+ct\omega\right)\cdot\omega\ d\omega = \frac{2c}{4\pi}\int_{|\omega|=1}\nabla\varphi_0\left(x\right)\cdot\omega\ d\omega = 0,$$

we have

$$\lim_{t\downarrow 0} v_t(x,t) = 0.$$

Theorem 5.5.1 *Suppose that $\varphi_0 \in C^3(\mathbb{R}^3)$ and $\varphi_1 \in C^2(\mathbb{R}^3)$. The Cauchy problem in (5.20) has a unique solution $u \in C^2\left(\mathbb{R}^3 \times \mathbb{R}^+\right)$ and this solution is given by*

$$u\left(x,t\right) = \frac{1}{4\pi c^2 t}\int_{|x-y|=ct}\varphi_1\left(y\right)d\sigma + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|x-y|=ct}\varphi_0\left(y\right)d\sigma\right).$$

This expression is known as Poisson's solution.

Exercise 90 *Let $u \in C^2\left(\bar{B}\right)$ with $B = \left\{x \in \mathbb{R}^3; |x| \le 1\right\}$ and define $\bar{u}$ as follows:*

$$\bar{u}\left(x\right) = \frac{1}{4\pi\left|x\right|^2}\int_{|y|=|x|}u(y)dy.$$

This $\bar{u}$ is called the spherical average of u. Show that:

$$\Delta\bar{u}\left(x\right) = \frac{1}{4\pi\left|x\right|^2}\int_{|y|=|x|}\Delta u(y)dy.$$

Hint: use spherical coordinates $x = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$. In these coordinates:

$$\Delta = r^{-2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Proof. To give a direct proof that the formula indeed gives a solution of the differential equation is quite tedious. Some key ingredients are Green's formula applied to the fundamental solution:

$$
\begin{aligned}
&-4\pi f(x) = \\
&= \int_{|z|<R} \frac{1}{|z|}\Delta_z f(x+z)dz - \frac{1}{R}\int_{|z|=R} \frac{\partial}{\partial r} f(x+z)d\sigma - \frac{1}{R^2}\int_{|z|=R} f(x+z)d\sigma,
\end{aligned}
$$

the observation that

$$\frac{\partial}{\partial t}\int_{|z|<ct} v(z)dz = c\int_{|z|=ct} v(z)d\sigma$$

and a version of the fundamental theorem of calculus on balls in $\mathbb{R}^3$:

$$
\begin{aligned}
&\int_{|z|<R} \frac{1}{|z|^2}\frac{\partial}{\partial |z|} g(z)\,dz = \int_{|\omega|=1}\int_{r=0}^{R} \frac{1}{r^2}\left(\frac{\partial}{\partial r} g(r\omega)\right) r^2 dr d\omega = \\
&= \int_{|\omega|=1} (g(R\omega)-g(0))\,d\omega = \int_{|z|=R} \frac{1}{|z|^2}(g(z)-g(0))\,d\sigma_z.
\end{aligned}
$$

Here we go for $u(x,t) = \frac{1}{t}\int_{|x-y|=ct} f(y)\,d\sigma$:

$$
\begin{aligned}
c^2\Delta u(x,t) &= \frac{c^2}{t}\int_{|z|=ct} \Delta_x f(x+z)d\sigma_z = c^3\int_{|z|=ct} \frac{1}{|z|}\Delta_z f(x+z)d\sigma_z = \\
&= c^2\frac{\partial}{\partial t}\int_{|z|<ct} \frac{1}{|z|}\Delta_z f(x+z)dz = \\
&= c^2\frac{\partial}{\partial t}\left(-4\pi f(x) + \frac{1}{ct}\int_{|z|=ct} \frac{\partial}{\partial |z|} f(x+z)d\sigma + \frac{1}{c^2t^2}\int_{|z|=ct} f(x+z)d\sigma\right) = \\
&= c^2\frac{\partial}{\partial t}\left(\int_{|z|=ct} \frac{1}{|z|}\frac{\partial}{\partial |z|} f(x+z)d\sigma + \int_{|z|=ct} \frac{1}{|z|^2} f(x+z)d\sigma\right) = \\
&= c^2\frac{\partial}{\partial t}\left(\int_{|z|=ct} \frac{1}{|z|^2}\frac{\partial}{\partial |z|}\Big(|z|\, f(x+z)\Big)\, dz\right) = \\
&= c\frac{\partial^2}{\partial t^2}\left(\int_{|z|<ct} \frac{1}{|z|^2}\frac{\partial}{\partial |z|}\Big(|z|\, f(x+z)\Big)\, dz\right) = \\
&= c\frac{\partial^2}{\partial t^2}\left(\int_{|z|=ct} \frac{1}{|z|} f(x+z)d\sigma\right) = \\
&= c\frac{\partial^2}{\partial t^2}\left(\frac{1}{ct}\int_{|z|=ct} f(x+z)d\sigma\right) = \frac{\partial^2}{\partial t^2} u(x,t).
\end{aligned}
$$

Indeed it solves $\frac{\partial^2}{\partial t^2}u(x,t) = c^2\Delta u(x,t)$. The initial conditions we have already checked above.

It remains to verify the uniqueness. Suppose that u_1 and u_2 are two different solutions to (5.20). Set $v = u_1 - u_2$ and consider the average of v over the sphere of radius r around some x

$$\bar{v}(r,t) = \frac{1}{4\pi r^2}\int_{|y-x|=r} v(y,t)dy.$$

Since
$$\frac{\partial^2}{\partial r^2}\Big(r\bar{v}\,(r,t)\Big) = r\frac{\partial^2}{\partial r^2}\bar{v}\,(r,t) + 2\frac{\partial}{\partial r}\bar{v}\,(r,t) = r\Delta\bar{v}\,(|y|\,,t)\,,$$
we may, by taking the average as in the last exercise, conclude that
$$\begin{aligned}\frac{\partial^2}{\partial r^2}\Big(r\bar{v}\,(r,t)\Big) &= \frac{r}{4\pi r^2}\int_{|y-x|=r}\Delta v(y,t)dy = \\ &= \frac{r}{4\pi c^2 r^2}\int_{|y-x|=r} v_{tt}(y,t)dy = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(r\bar{v}\,(r,t)\right).\end{aligned}$$
Since $v(\cdot,0)$ and $v_t(\cdot,0)$ are both 0 also $r\bar{v}\,(r,0)$ and $r\bar{v}_t\,(r,0)$ are zero implying that $r\bar{v}\,(t,r) = 0$. Moreover, for all $t \geq 0$ one finds $(r\bar{v}\,(t,r))_{|r=0} = 0$. So we may use the version of d'Alembert formula we derived for $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ in Exercise 69. We find that $r\bar{v}\,(t,r) = 0$. So the average of $u_1(\cdot,t)$ and $u_2(\cdot,t)$ are equal over each sphere in $\mathbb{R}^3$ for all $t > 0$. This contradicts the assumption that $u_1 \neq u_2$. ■

Exercise 91 *Writing out Poisson's solution one obtains Kirchhoff's formula. From the cited books (see the bibliography) the following versions of Kirchhoff's formula have been copied:*

1. $$u(x,t) = \frac{1}{c^2}⨍_{\partial B(x,t)} th(y) + g(y) + Dg(y)\cdot(y-x)\ dS(y)$$

where $⨍_A v(y)dy = \left(\int_A 1dy\right)^{-1}\int_A v(y)dy.$

2. $$u(x,t) = \frac{1}{4\pi c^2 t^2}\int_{|x-y|=ct}\left[t\psi(y) + \varphi(y) + \nabla\varphi\cdot(x-y)\right]d\sigma$$

3. $$u(x,t) = \frac{1}{4\pi c^2 t^2}\int_{|x-y|=ct}\left[tg(y) + f(y) + \sum_i f_{y_i}(y)(y_i - x_i)\right]d\sigma$$

4. $$\begin{aligned}u(x,t) &= \frac{1}{4\pi}\iint\limits_S\left[\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial u_1}{\partial n} - \frac{2}{cr}\frac{\partial r}{\partial n}\frac{\partial u_1}{\partial t_1}\right]_{t_1=0} dS + \\ &+ \frac{1}{4\pi}\iiint\limits_\Omega \frac{1}{c^2 r}F\left(y, t-\frac{r}{c}\right)dy\end{aligned}$$

5. $$u\,(x,t) = \frac{\partial}{\partial t}t\omega[f;x,t] + t\omega[g;x,t]$$

where the following expression is called Kirchhoff's solution:

$$t\omega[g;x,t] = \\ \frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} g(x_1 + t\sin\theta\cos\phi, x_2 + t\sin\theta\sin\phi, x_3 + t\cos\theta)\sin\theta d\theta d\phi.$$

Explain which boundary value problem each of these u solve (if that one does) and what the appearing symbols mean.

5.5.2 2 space dimensions

In 2 dimensions no transformation to a 1 space dimensional problem exists as in 3 dimensions. However, having a solution in 3 space dimensions one could just use that formula for initial values that do not depend on x_3. The solution that comes out should not depend on the x_3 variable and hence

$$u_{tt} = c^2 \left(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} \right) = c^2 \left(u_{x_1x_1} + u_{x_2x_2} \right).$$

Borrowing the formula from Theorem 5.5.1 we obtain, with $y = (y_1, y_2)$

$$\begin{aligned} & \frac{1}{4\pi c^2 t} \int_{|(x,0)-(y,y_3)|=ct} \varphi_1(y_1,y_2)\, d\sigma = \\ = \; & \frac{2}{4\pi c^2 t} \int_{|x-y|\le ct} \varphi_1(y_1,y_2) \frac{ct}{\sqrt{c^2t^2 - (x_1-y_1)^2 - (x_2-y_2)^2}} dy_1 dy_2 \\ = \; & \frac{1}{2\pi c} \int_{|x-y|\le ct} \frac{\varphi_1(y)}{\sqrt{c^2t^2 - |x-y|^2}} dy. \end{aligned}$$

So we may conclude that

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \\ u(x,0) = \phi_0(x) & \text{for } x \in \mathbb{R}^2, \\ u_t(x,0) = \phi_1(x) & \text{for } x \in \mathbb{R}^2. \end{cases} \tag{5.22}$$

can be solved uniquely. This idea to go down from a known solution method in dimension n in order to find a solution in dimension $n-1$ is called the method of descent.

Theorem 5.5.2 *Suppose that $\varphi_0 \in C^3(\mathbb{R}^2)$ and $\varphi_1 \in C^2(\mathbb{R}^2)$. The Cauchy problem in (5.22) has a unique solution $u \in C^2\left(\mathbb{R}^2 \times \mathbb{R}^+\right)$ and this solution is given by*

$$\begin{aligned} u(x,t) \; = \; & \frac{1}{2\pi c} \int_{|x-y|\le ct} \frac{\varphi_1(y)}{\sqrt{c^2t^2 - |x-y|^2}} dy + \\ & + \frac{\partial}{\partial t} \left(\frac{1}{2\pi c} \int_{|x-y|\le ct} \frac{\varphi_0(y)}{\sqrt{c^2t^2 - |x-y|^2}} dy \right). \end{aligned}$$

This expression is known as Poisson's solution in 2 space dimensions.

As a result of this 'spreading out' higher frequencies die out faster then lower frequencies in 2 dimensions. Flatlanders should be baritones.

Bibliography

[1] H. Brezis: Analyse Fonctionnelle, Masson, 1983

[2] E. Coddington and N. Levinson: Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

[3] R. Courant and D. Hilbert: Methods of Mathematical Physics I, Interscience, 1953.

[4] R. Courant and D. Hilbert: Methods of Mathematical Physics II, Interscience, 1961.

[5] L.C. Evans, Partial Differential Equations, AMS, 1998.

[6] B. Dacorogna: Direct methods in the calculus of variations, Springer, 1993.

[7] E. DiBenedetto: Partial Differential Equations, Birkhäuser, 1995.

[8] P.R. Garabedian: Partial Differential Equations, AMS Chelsea Publishing, 1964.

[9] M. Giaquinta and S. Hildebrandt: Calculus of Variations I, Springer, 1996

[10] M. Giaquinta and S. Hildebrandt: Calculus of Variations II, Springer, 1996

[11] Fritz John: Partial Differential Equations 4^{th} edition, Springer, 1981.

[12] S.L. Sobolev: Partial Differential Equations of Mathematical Physics, Pergamon Press, 1964 (republished by Dover in 1989).

[13] W.A. Strauss: Partial Differential Equations, an introduction, Wiley, 1992.

[14] W. Walter: Gewöhnliche Differentialgleichungen, Springer

[15] H.F. Weinberger: A first course in Partial Differential Equations, Blaisdell, 1965 (republished by Dover 1995).

For ordinary differential equations see [2] or [14].
For derivation of models see [3], [4], [7, Chapter 0], [8].
For models arising in a variational setting see [9] and [10].
Introductionary text in p.d.e.: [13] and [15].
Modern graduate texts in p.d.e.: [5] and [7].
Direct methods in the calculus of variations: [5, Chapter 8] and [6].
For Cauchy-Kowalevski see [11].
For the functional analytic tools see [1].