EMBEDDING RESULTS FOR γ-SPACES

MARK VERAAR

Abstract. In this note we consider embeddings between function spaces and the space of γ-radonifying operators. We provide some positive and negative results and discuss the connections with type and cotype.

1. Introduction

The γ-spaces (or sometimes called spaces of γ-radonifying operators) play an important role in functional analysis (see the recent survey [25] for a details). During the last decade the unpublished manuscript [17] on functional calculus has played an influential role in new applications of the theory of γ-radonifying operators to different parts of functional analysis. In vector-valued harmonic analysis and operator theory the γ-spaces have been used in [12, 13, 15, 14, 17, 18, 19]. In vector-valued stochastic analysis it has turned out that γ-spaces provide the right framework to obtain equivalent norms for stochastic integrals (see [27] and references therein) and this has let to a rapid further development (see [5, 22, 23, 26, 30, 28]).

In [16] embedding results between certain X-valued Besov spaces and γ-spaces have been studied under type and cotype assumptions on the Banach space X. This immediately has led to applications in stochastic evolution equations in [30]. Let us mention that embeddings for γ-spaces have also been used in [4] for invariant measures and ergodic theory, in [6] for equations with fractional noise, in [24] for asymptotic behavior of semigroups.

In this note we discuss possible improvements for the embedding results for γ-spaces and we provide both positive and negative results. The results in this note have already found applications in [6]: it can be used to obtain sharp sufficient condition for stochastic integrability with respect to fractional Brownian motion. Another application can be found in [29]. Here together with van Neerven and Weis we have consider maximal regularity results for parabolic problems in γ-spaces. Using embedding results below one can derive regularity results in terms of more classical spaces.

Acknowledgement. The author thanks the anonymous referee for carefully reading the manuscript.

2. Preliminaries

2.1. **Type and cotype.** A Banach space X is said to have *type* p with $p \in [1, 2]$ if there exists a constant $C \geq 0$ such that for all finite sequences $x_1, \ldots, x_N$ in X we

The author is supported by VENI subsidy 639.031.930 of the Netherlands Organisation for Scientific Research (NWO).

have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^p\Big)^{\frac1p} \leq C\Big(\sum_{n=1}^N \|x_n\|^p\Big)^{\frac1p}.$$

The space X is said to have *cotype* q with $q \in [2,\infty]$ if there exists a constant $C \geq 0$ such that for all finite sequences $x_1, \ldots, x_N$ in X we have

$$\Big(\sum_{n=1}^N \|x_n\|^q\Big)^{\frac1q} \leq C\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n x_n\Big\|^q\Big)^{\frac1q},$$

where one needs to replace the left-hand side by $\sup_{1\leq n\leq N}\|x_n\|$ if $q=\infty$. For details on type and cotype we refer to the monographs [1, 8]. Note that for all $p \in [1,\infty)$, L^p has type $p \wedge 2$ and cotype $p \vee 2$.

2.2. **γ-spaces.** We briefly recall the definition of γ-radonifying operators. For details we refer to [8, Chapter 12] and [25]. Let H be a separable Hilbert space with an orthonormal basis $(h_n)_{n\geq1}$. Let $(\gamma_n)_{n\geq1}$ be a sequence of independent real-valued standard Gaussian random variables. For a bounded operator $R: H \to X$ we say that R is a *γ-radonifying operator* if $\sum_{n\geq1}\gamma_n R h_n$ converges in $L^2(\Omega;X)$ and in this case let

$$\|R\|^2_{\gamma(H,X)} := \mathbb{E}\Big\|\sum_{n\geq1}\gamma_n R h_n\Big\|^2.$$

The space $\gamma(H,X)$ is a Banach space and is called the *γ-space*.

Let $(A,\mathcal{A},\nu)$ be a σ-finite measure space. For a strongly ν-measurable function $f: A\to X$ such that for all $x^*\in X^*$, $\langle f, x^*\rangle$ belongs to $L^2(A)$ one can define a Pettis integral operator $I_f: L^2(A)\to X$ by $I_f g = \displaystyle\int_A fg\,d\mu$. In the sequel I_f and f will be identified and we will also write $\|f\|_{\gamma(L^2(A),E)} := \|I_f\|_{\gamma(L^2(A),X)}$. The simple functions $f: A\to X$ are dense in $\gamma(L^2(A),X)$.

In a similar way as in [31] one can show that for a Banach function space X with finite cotype, one has $\gamma(L^2(A),X) = X(L^2(A))$ and there is a $C>0$ such that for all $f: A\to X$ with $\|f\|_{X(L^2(A))} := \Big(\int_A |f|^2\,d\nu\Big)^{1/2} < \infty$ one has $f\in\gamma(L^2(A),X)$ and

$$C^{-1}\|f\|_{X(L^2(A))} \leq \|f\|_{\gamma(L^2(A),X)} \leq C\|f\|_{X(L^2(A))}. \tag{2.1}$$

2.3. **Vector-valued function spaces.** The Fourier transform of a function $f\in L^1(\mathbb{R}^d;X)$ will be normalized as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx, \quad \xi\in\mathbb{R}^d.$$

We briefly recall the definition of different types of vector-valued function spaces. For details we refer to [36, 37] for the scalar case and [3, 11, 33, 34, 35, 38] for the vector-valued case.

Let $\phi\in\mathscr{S}(\mathbb{R}^d)$ be a fixed Schwartz function whose Fourier transform $\widehat{\phi}$ is nonnegative and has support in $\{\xi\in\mathbb{R}^d : \frac12 \leq |\xi| \leq 2\}$ and which satisfies $\sum_{k\in\mathbb{Z}}\widehat{\phi}(2^{-k}\xi) = 1$ for $\xi\in\mathbb{R}^d\setminus\{0\}$. Define the sequence $(\varphi_k)_{k\geq0}$ in $\mathscr{S}(\mathbb{R}^d)$ by $\widehat{\varphi_k}(\xi) = \widehat{\phi}(2^{-k}\xi)$ for $k\geq1$ and $\widehat{\varphi_0}(\xi) = 1-\sum_{k\geq1}\widehat{\varphi_k}(\xi)$ for $\xi\in\mathbb{R}^d$.

For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ the *Besov space* $B^s_{p,q}(\mathbb{R}^d; X)$ is defined as the space of all X-valued tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d;X)} := \Big(\sum_{k\ge 0} \|2^{ks}\varphi_k * f\|^q_{L^p(\mathbb{R}^d;X)}\Big)^{1/q}$$

is finite. This space is a Banach space and a different function φ yields an equivalent norm.

For $p \in [1, \infty)$, $1 \le q \le \infty$ and $s \in \mathbb{R}$ the *Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^d; X)$ is defined as the space of all X-valued tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d;X)} := \Big\|\Big(\sum_{k\ge 0} 2^{ks}\|\varphi_k * f\|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$

is finite. This space is a Banach space and a different function φ yields an equivalent norm. Note that $F^s_{p,p}(\mathbb{R}^d; X) = B^s_{p,p}(\mathbb{R}^d; X)$.

For $p \in [1, \infty]$ and $s \in \mathbb{R}$ the *Bessel-potential space* $H^s_p(\mathbb{R}^d; X)$ is defined as the space of all X-valued tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{H^s_p(\mathbb{R}^d;X)} := \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2}\widehat{f}]\|_{L^p(\mathbb{R}^d;X)} < \infty.$$

Recall the following assertion (see [34, 35]): for all $s \in \mathbb{R}$ and $p \in (1, \infty)$ one has

$$B^s_{p,1}(\mathbb{R}^d; X) \hookrightarrow F^s_{p,1}(\mathbb{R}^d; X) \hookrightarrow H^s_p(\mathbb{R}^d; X) \hookrightarrow F^s_{p,\infty}(\mathbb{R}^d; X) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^d; X). \tag{2.2}$$

Also recall from [34] that for $1 < p_0 < p_1 < \infty$, $q_0, q_1 \in [1, \infty]$ and $s_0 - \frac{d}{p_0} \ge s_1 - \frac{d}{p_0}$ one has

$$F^{s_0}_{p_0,q_0}(\mathbb{R}^d; X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d; X), \quad \text{and} \quad H^{s_0}_{p_0}(\mathbb{R}^d; X) \hookrightarrow H^{s_1}_{p_1}(\mathbb{R}^d; X). \tag{2.3}$$

For UMD spaces X there is an equivalent norm on $H^s_p(\mathbb{R}^d; X)$ (see [7] for details on UMD spaces).

Proposition 2.1. *Let X be a UMD space, $p \in (1, \infty)$ and $s \in \mathbb{R}$. Then $f \in H^s_p(\mathbb{R}^d; X)$ if and only if*

$$\|f\|_{F^{s,\varphi}_{p,rad}(\mathbb{R}^d;X)} := \sup_{n\ge 1} \Big\|\sum_{k=0}^n r_k 2^{sk}\varphi_k * f\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} < \infty.$$

Moreover, $\sum_{k\ge 0} r_k 2^{sk}\varphi_k * f$ converges in $L^p(\Omega \times \mathbb{R}^d; X)$ and there is a $C > 0$ not depending on f such that

$$C^{-1}\|f\|_{F^{s,\varphi}_{p,rad}(\mathbb{R}^d;X)} \le \|f\|_{H^s_p(\mathbb{R}^d;X)} \le C\|f\|_{F^{s,\varphi}_{p,rad}(\mathbb{R}^d;X)}.$$

As in the real case (see [36, Section 2.3.3] the result is an easy consequence of the vector-valued Mihlin multiplier theorem [39, Proposition 3]. Note that using [20, Theorem 9.29]) one sees that $\|f\|_{F^{s,\varphi}_{p,\mathrm{rad}}(\mathbb{R}^d;X)} = \Big\|\sum_{k\ge 0} r_k 2^{sk}\varphi_k * f\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}$ if X does not contain a copy of c_0.

Notation. The constants C we use may differ from line to line.

3. Results

3.1. **Embeddings for Bessel-potential spaces.** Under geometric conditions on X one can improve the embeddings in (2.2).

Proposition 3.1. *Let X be a UMD Banach space and let $p \in (1,\infty)$ and $s \in \mathbb{R}$. Let $p_0 \in (1,2]$, $q_0 \in [2,\infty)$.*

(1) *If X has type p_0, then*
$$B^s_{p,p\wedge p_0}(\mathbb{R}^d;X) \hookrightarrow F^s_{p,p_0}(\mathbb{R}^d;X) \hookrightarrow H^s_p(\mathbb{R}^d;X).$$
(2) *If X has cotype $q_0 \in [2,\infty)$, then*
$$H^s_p(\mathbb{R}^d;X) \hookrightarrow F^s_{p,q_0}(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee q_0}(\mathbb{R}^d;X),$$

Proof. Since the proofs of (1) and (2) are similar we only provide details for (1). Let $f \in \mathscr{S}'(\mathbb{R}^d;X)$. By Proposition 2.1, [20, Theorem 4.7] and type p_0 one has
$$\begin{aligned}\|f\|_{H^s_p(\mathbb{R}^d;X)} &\leq C \sup_{n\geq 1} \Big\| \sum_{k=0}^n r_k 2^{sk}\varphi_k * f\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &\leq C \sup_{n\geq 1} \Big(\int_{\mathbb{R}^d} \Big\| \sum_{k=0}^n r_k 2^{sk}\varphi_k * f\Big\|^p_{L^{p_0}(\Omega;X)}\,dx\Big)^{1/p} \\ &\leq C \sup_{n\geq 1} \Big(\int_{\mathbb{R}^d} \Big(\sum_{k=0}^n \|2^{sk}\varphi_k * f\|^{p_0}\Big)^{p/p_0}\,dx\Big)^{1/p} = C\|f\|_{F^s_{p,p_0}(\mathbb{R}^d;X)}.\end{aligned}$$
If $p \leq p_0$, then $\|f\|_{F^s_{p,p_0}(\mathbb{R}^d;X)} \leq \|f\|_{F^s_{p,p}(\mathbb{R}^d;X)} = \|f\|_{B^s_{p,p}(\mathbb{R}^d;X)}$. If $p > p_0$, then $\|f\|_{F^s_{p,p_0}(\mathbb{R}^d;X)} \leq \|f\|_{B^s_{p,p_0}(\mathbb{R}^d;X)}$ due to Minkowksi's inequality (or the triangle inequality in L^{p/p_0}). ☐

Remark 3.2. It would be interesting to know whether type p_0 and cotype q_0 are necessary in Proposition 3.1.

Note that if X is a Hilbert space or $X = \mathbb{R}$, Proposition 3.1 reduces to $F^s_{p,2}(\mathbb{R}^d;X) = H^s_p(\mathbb{R}^d;X)$ and the embeddings (see [36, Remark 2.3.3.4])
$$B^s_{p,p\wedge 2}(\mathbb{R}^d;X) \hookrightarrow H^s_p(\mathbb{R}^d;X) \hookrightarrow B^s_{p,p\vee 2}(\mathbb{R}^d;X).$$

3.2. **Embeddings for $\gamma(L^2(\mathbb{R}^d),X)$.** The following result was proved in [32]. Here the measure space $\mathbb{R}^d$ can be replaced by a more general measure space.

Proposition 3.3. *Let X be a Banach space. Then the following assertions hold:*

(1) *X has type 2 if and only if $L^2(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$.*
(2) *X has cotype 2 if and only if $\gamma(L^2(\mathbb{R}^d);X) \hookrightarrow L^2(\mathbb{R}^d;X)$.*

The following result was obtained in [16].

Proposition 3.4. *Let X be a Banach space and let $1 \leq p \leq 2 \leq q \leq \infty$.*

(1) *X has type p if and only if $B^{(\frac1p-\frac12)d}_{p,p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$.*
(2) *X has cotype q if and only if $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{(\frac1q-\frac12)d}_{q,q}(\mathbb{R}^d;X)$.*

Actually the proof in [16] shows that $B^{(\frac1p-\frac12)d}_{p,r}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$ implies that X has type r, and $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow B^{(\frac1q-\frac12)d}_{q,r}(\mathbb{R}^d;X)$ implies X has cotype r. Therefore in the Besov scale one cannot improve on the embeddings of Proposition

3.4 through the microscopic parameter r. For the Triebel-Lizorkin and Bessel-potential spaces this is different as will turn out from the following result.

Proposition 3.5. *Let X be a Banach space and let $1 \leq p \leq 2 \leq q < \infty$.*

(1) *If X has type p then for all $p_0 \in [1, p)$ and all $r \in [1, \infty]$ one has*

$$F_{p_0,r}^{(\frac{1}{p_0}-\frac{1}{2})d}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X),$$

and for all $p_0 \in (1, p)$ one has

$$H_{p_0}^{(\frac{1}{p_0}-\frac{1}{2})d}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X).$$

(2) *If X has cotype q, then for all $q_0 \in (q, \infty)$ and all $r \in [1, \infty]$ one has*

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow F_{q_0,r}^{(\frac{1}{q_0}-\frac{1}{2})d}(\mathbb{R}^d; X)$$

and for all $q_0 \in (q, \infty)$ one has

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H_{q_0}^{(\frac{1}{q_0}-\frac{1}{2})d}(\mathbb{R}^d; X)$$

Proof.

(1): Let $1 \leq p_0 < p$. Let $s = (\frac{1}{p} - \frac{1}{2})d$ and $s_0 = (\frac{1}{p_0} - \frac{1}{2})d$. By [34, Theorem 5] (see [37, Theorem 2.7.1] for the scalar case) one has

$$F_{p_0,r}^{s_0}(\mathbb{R}^d; X) \hookrightarrow F_{p,p}^{s}(\mathbb{R}^d; X) = B_{p,p}^{s}(\mathbb{R}^d; X).$$

Now the result for F-spaces follows from Proposition 3.4. The result for H follows from (2.2).

(2): This can be proved in a similar way. □

It is natural to ask for what values of r the embeddings in Proposition 3.5 hold for the limiting exponents $p_0 = p$ and $q_0 = q$. From Proposition 3.4 and the fact that $F_{p,r}^{s}$ increases in the parameter r one sees that (1) holds for the limiting case $p = p_0$ if $r \in [1, p]$. Similarly, (2) holds for the limiting case $q = q_0$ if $r \in [p, \infty]$.

In the following remark we explain that one cannot take a better r in the cases $p = 2$ and $q = 2$ in the F-scale.

Remark 3.6.

(1) If $p = 2$, then one cannot take $p_0 = 2$ in the embeddings for the F-spaces in Proposition 3.5. Indeed, already in the scalar case $X = \mathbb{R}$ one has $\gamma(L^2(\mathbb{R}^d), \mathbb{R}) = L^2(\mathbb{R}^d) = F_{2,2}^0(\mathbb{R}^d)$. It is well-known that $F_{2,r}^0(\mathbb{R}^d)$ is strictly larger then $F_{2,2}^0(\mathbb{R}^d)$ if $r > 2$. Similarly, if $q = 2$, one cannot take $q_0 = 2$ in the embeddings for the F-spaces since $F_{2,r}^0(\mathbb{R}^d)$ is strictly smaller then $F_{2,2}^0(\mathbb{R}^d)$ if $r < 2$.

(2) If $p_0 = p = 2$ in Proposition 3.5, then one has $H_2^0(\mathbb{R}^d; X) = L^2(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X)$ by Proposition 3.3. Similarly if $q_0 = q = 2$, then $\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow L^2(\mathbb{R}^d; X) = H_2^0(\mathbb{R}^d; X)$.

In the cases $1 < p < 2$ and $2 < q < \infty$ it remains unclear whether Proposition 3.5 holds with $p_0 = p$ and $q_0 = q$.

In the following result we provide a negative answer for $r = \infty$.

Proposition 3.7. *Let X be a Banach space. Let $p \in [1,2)$. If*

$$F^{\frac{1}{p}-\frac{1}{2}}_{p,\infty}(\mathbb{R};X) \hookrightarrow \gamma(L^2(\mathbb{R}),X), \tag{3.1}$$

then there is an $\varepsilon > 0$ such that X has type $p+\varepsilon$.

Remark 3.8. In particular, Proposition 3.7 shows that the embedding (3.1) does not hold for spaces $X = L^p(0,1)$ or $X = \ell^p$ with $p \in [1,2)$.

Proof. Let $s = \frac{1}{p} - \frac{1}{2}$. Recall from [34] (see [37, 2.5.10] for the scalar case) that the following is an equivalent norm on $F^s_{p,\infty}(\mathbb{R};X)$: $|||f|||_{F^s_{p,\infty}(\mathbb{R};X)} = \|f\|_{L^p(\mathbb{R};X)} + [f]_{F^s_{p,\infty}(\mathbb{R};X)}$, where

$$[f]_{F^s_{p,\infty}(\mathbb{R};X)} = \Big\| x \mapsto \sup_{t>0} t^{-s-1} \int_{|h|\leq t} \Delta_h(f)(x)(\cdot) \Big\|_{L^p(\mathbb{R})}$$

with $\Delta_h(f)(x) = \|f(x+h) - f(x)\|_X$, $x \in \mathbb{R}^d$. Note that for a function $f \in C^s(\mathbb{R};X)$ with support in $[0,1]$ one has $\|f\|_{L^p(\mathbb{R};X)} \leq \|f\|_{L^\infty(\mathbb{R})}$. Furthermore, for all $t > 0$ one has

$$t^{-s-1} \int_{|h|\leq t} \Delta_h(f)(x) \leq t^{-s} M\big(\|f(\cdot)\|\big)(x) + t^{-s}\|f(x)\|_X,$$

where Mg denotes the Hardy-Littlewood maximal function of $g : \mathbb{R} \to R$. It follows that the part $\sup_{t>1}$ can be estimated as

$$\Big\| \sup_{t>1} t^{-s-1} \int_{|h|\leq t} \Delta_h(f)(x) \Big\|_{L^p(\mathbb{R})} \leq \Big\| M\big(\|f(\cdot)\|\big) + \|f\| \Big\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R};X)},$$

where we used the L^p-boundedness of M. For the part $\sup_{t\in(0,1]}$ note that $\Delta_h(f)(x) \leq [f]_{C^s(\mathbb{R};X)}|h|^s$ and $\operatorname{supp.}\big(\Delta_h(f)(\cdot)\big) \subseteq [-1,2]$, and therefore,

$$\begin{aligned}
&\Big\| \sup_{t\in(0,1]} t^{-s-1} \int_{|h|\leq t} \Delta_h(f)(x) \Big\|_{L^p(\mathbb{R})} \\
&= \Big(\int_{\mathbb{R}} \Big(\sup_{t\in(0,1]} t^{-s-1} \int_{|h|\leq t} [f]_{C^s(\mathbb{R};X)} |h|^s \mathbf{1}_{[-1,2]}(x)\, dh \Big)^p dx \Big)^{1/p} = \frac{2 \cdot 3^{1/p} [f]_{C^s(\mathbb{R};X)}}{s+1}.
\end{aligned}$$

Therefore, we obtain that for some constant K one has

$$|||f|||_{F^s_{p,\infty}(\mathbb{R};X)} \leq K([f]_{C^s(\mathbb{R};X)} + \|f\|_{L^\infty(\mathbb{R})}) =: K\|f\|_{C^s(\mathbb{R};X)},$$

for all $f \in C^s(\mathbb{R};X)$ with support in $[0,1]$. Since for any function $f \in C^s([0,1];X)$ which satisfies $f(0) = f(1) = 0$ one has $\|f\|_{C^s(\mathbb{R};X)} \leq \|f\|_{C^s([0,1];X)}$, from (3.1) we obtain $C^s([0,1];X) \hookrightarrow \gamma(L^2(0,1),X)$. Now [26, Theorem 5.2] yields the required result. □

Next we obtain an embedding result for H-spaces with values in a p-convex or q-concave Banach lattice. It improves Proposition 3.4. For details on Banach lattices we refer the reader to [21]. For sake of completeness let us recall the definition of p-convexity and q-concavity for $p, q \in [1,\infty)$. A Banach lattice X is called *p-convex* if there is a constant $C > 0$ such that for all $n \geq 1$ and for all $x_1, \ldots, x_n \in X$ one has

$$\Big\| \Big(\sum_{i=1}^n |x_i|^p \Big)^{1/p} \Big\| \leq C \Big(\sum_{i=1}^n \|x_i\|^p \Big)^{1/p}.$$

A Banach lattice X is called *q-concave* if there is a constant $C > 0$ such that for all $n \geq 1$ and for all $x_1, \ldots, x_n \in X$ one has

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \leq C\Big\|\Big(\sum_{i=1}^{n} |x_i|^q\Big)^{1/q}\Big\|.$$

Recall that every p-convex Banach lattice has type $p \wedge 2$ and every q-concave Banach lattice has cotype $q \vee 2$. In [10] it has been shown that the Lorentz space $L^{p,p'}$ with $p \in (1,2)$ has type p, but is not p-convex.

Theorem 3.9. *Let X be a Banach lattice with finite cotype and let $1 \leq p \leq 2 \leq q < \infty$.*

(1) *If X is p-convex, then*

$$H_p^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d; X) \hookrightarrow \gamma(L^2(\mathbb{R}^d), X). \tag{3.2}$$

(2) *If X is q-concave then*

$$\gamma(L^2(\mathbb{R}^d), X) \hookrightarrow H_q^{(\frac{1}{q}-\frac{1}{2})d}(\mathbb{R}^d; X). \tag{3.3}$$

Remark 3.10.

(1) From Proposition 3.1 one sees that under the assumption that X has UMD, Theorem 3.9 indeed improves Proposition 3.4.
(2) As in [16] one can deduce embedding results for $\gamma(H_2^{\alpha}(\mathbb{R}^d), X)$.
(3) As in [16] one obtains embedding result for $\gamma(L^2(D), X)$ where $D \subseteq \mathbb{R}^d$.

Proof. Since X has cotype q, it does not contain a copy of c_0 or ℓ^∞. Therefore, by [21, Theorem 1.a.5 and Proposition 1.a.7] X is order continuous.

To prove (1) or (2) it suffices by a density argument to consider one fixed $f \in L^p(\mathbb{R}^d) \otimes X$ or $f \in L^q(\mathbb{R}^d) \otimes X$. We show how one can reduce the proof of (1) and (2) to the case where X is a Banach function space. Write $f = \sum_{n=1}^N f_n \otimes x_n$. Let $x = \sum_{n=1}^N |x_n|$. Then x is a weak unit for the band $X_x = \{y \in X : |y| = \sup_{n\geq 1} x \wedge n|y|\}$ (see [2, p. 39]). Since X_x is a closed sublattice of X (see [2, Theorem 3.46]), it is also order complete. Therefore, [21, Theorem 1.b.14] implies that X_x is order isometric to a Banach function space.

(1): By the above remarks we may assume that X is a p-convex Banach function space. We first claim that there is a constant C such that for all $f \in L^p(A) \otimes X$, one has

$$\|f\|_{X(L^p(A))} \leq C\|f\|_{L^p(A;X)}. \tag{3.4}$$

We first prove (3.4) in the case $f : A \to X$ is a simple function. We can write $f = \sum_{k=1}^K \mathbf{1}_{A_k} x_k$ with $A_1, \ldots, A_K \in \mathcal{A}$ disjoint and $\mu(A_k) < \infty$ and $x_k \in X$ for all $1 \leq k \leq K$. Since X is p-convex one has

$$\|f\|_{X(L^p(A))} = \Big\|\Big(\sum_{k=1}^K \mu(A_k)|x_k|^p\Big)^{1/p}\Big\| \leq C\Big(\sum_{k=1}^K \mu(A_k)\|x_k\|^p\Big)^{1/p} = C\|f\|_{L^p(A;X)}.$$

If $f \in L^p(A) \otimes X$, we can write $f = \sum_{n=1}^N f_n \otimes x_n$ with $f_1, \ldots, f_N \in L^p(A)$ and $x_1, \ldots, x_N \in X$. We can find simple functions $(f_n^{(j)})_{j\geq 1}$ in $L^p(A)$ such that

$\lim_{j\to\infty} f_n^{(j)} = f_n$ in $L^p(A)$. Let $f^{(j)} = \sum_{n=1}^N f_n^{(j)} x_n$. Then $\lim_{j\to\infty} f^{(j)} = f$ both in $X(L^p(A))$ and in $L^p(A;X)$. Now the estimate (3.4) follows from

$$\|f\|_{X(L^p(A))} \leq \lim_{j\to\infty} \|f^{(j)}\|_{X(L^p(A))} \leq C \lim_{j\to\infty} \|f^{(j)}\|_{L^p(A;X)} = C\|f\|_{L^p(A;X)}.$$

To prove (1) denote $\alpha = (\frac{1}{p} - \frac{1}{2})d$. Let (S,σ,μ) be the underlying σ-finite measure space of the Banach function space. By density it suffices to consider $f \in \mathscr{S}(\mathbb{R}^d) \otimes X$. Let $f_1,\ldots,f_N \in \mathscr{S}(\mathbb{R}^d)$ and $x_1,\ldots,x_N \in X$ be such that $f = \sum_{n=1}^N f_n \otimes x_n$. As short-hand notation we write $f_s = \sum_{n=1}^N f_n x_n(s) \in \mathscr{S}(\mathbb{R}^d)$ for $s \in S$. From (2.1), (2.3) and (3.4) it follows that

$$\begin{aligned}\|f\|_{\gamma(L^2(\mathbb{R}^d),X)} &\leq C\Big\|s \mapsto \|f_s\|_{L^2(\mathbb{R}^d)}\Big\|_X \leq C\Big\|s \mapsto \|f_s\|_{H_p^\alpha(\mathbb{R}^d)}\Big\|_X \\ &= C\Big\|s \mapsto \|(1-\Delta)^{\alpha/2} f_s\|_{L^p(\mathbb{R}^d)}\Big\|_X \\ &\leq C\|s \mapsto (1-\Delta)^{\alpha/2} f_s\|_{L^p(\mathbb{R}^d;X)} = C\|f\|_{H_p^\alpha(\mathbb{R}^d;X)}.\end{aligned}$$

(2) : This can be proved in a similar way. Again we may assume that X is a q-concave Banach function space. Arguing as before one can show that there is a constant C such that for all $f \in L^q(A) \otimes X$, one has $\|f\|_{L^q(A;X)} \leq C_q \|f\|_{X(L^q(A))}$. To prove the required embedding $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow H_q^{\frac{1}{q}-\frac{1}{2}}(\mathbb{R}^d,X)$ it suffices to consider simple functions $f : \mathbb{R}^d \to X$. Now one can proceed as in step (1). □

Remark 3.11. In the converse direction one can show the following results:

1. If $F_{p,1}^{(\frac{1}{p}-\frac{1}{2})d}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$, then X has type p.
2. If $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow F_{q,\infty}^{(\frac{1}{q}-\frac{1}{2})d}(\mathbb{R}^d;X)$ then X has cotype q.

However, the proof is rather technical and we choose not to present it here. In particular, it follows from (1) and (2.2) that the embedding (3.2) implies type p. Similarly (3.3) implies cotype q. Note that if $d = 1$, one can deduce (1) from (2.2) and [16, Theorem 3.3], where it is shown that $B_{p,1}^{\frac{d}{p}-\frac{d}{2}}(0,1;X) \hookrightarrow \gamma(L^2(0,1),X)$ implies that X has type p.

Finally, we present two results which one can combine with Theorem 3.9 to obtain new embedding result. The proof of the first result is left to the reader.

Proposition 3.12. *Let $p \in [1,\infty)$ and let X and Y be Banach spaces.*

1. *If Y is isomorphic to a closed subspace of X and $H_p^s(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X)$, then $H_p^s(\mathbb{R}^d;Y) \hookrightarrow \gamma(L^2(\mathbb{R}^d),Y)$*
2. *If Y is isomorphic to a closed subspace of X and $\gamma(L^2(\mathbb{R}^d),X) \hookrightarrow H_p^s(\mathbb{R}^d;X)$, then $\gamma(L^2(\mathbb{R}^d),Y) \hookrightarrow H_p^s(\mathbb{R}^d;Y)$*

The same holds with H_p^s replaced by $F_{p,q}^s$ or $B_{p,q}^s$ with $q \in [1,\infty]$.

Proposition 3.13. *Let X be a UMD space. Let $p,p' \in (1,\infty)$ be such that $\frac{1}{p}+\frac{1}{p'} = 1$ and let $s \in \mathbb{R}$. One has*

$$H_p^s(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}^d),X) \quad \text{if and only if} \quad \gamma(L^2(\mathbb{R}^d),X^*) \hookrightarrow H_{p'}^{-s}(\mathbb{R}^d;X^*).$$

The same holds with H_p^s replaced by $F_{p,q}^s$ or $B_{p,q}^s$ with $q \in (1,\infty)$.

Proof. Since X is a UMD space it is reflexive and therefore by [9, Corollary III.2/12] and [9, Theorem IV.1/1] one can obtain that $H^s_p(\mathbb{R}^d;X)^* = H^{-s}_{p'}(\mathbb{R}^d;X^*)$. Since X has UMD it is K-convex and therefore one has $\gamma(L^2(\mathbb{R}^d),X)^* = \gamma(L^2(\mathbb{R}^d),X^*)$ (see [25, Theorem 10.9]). Now the result follows from a duality argument. □

Finally, we formulate two open problems which we state in dimension one. The corresponding dual statements are open as well.

Problem 3.14. *Let $p \in [1,2)$. Characterize those Banach spaces X and numbers $r \in (p,\infty)$ for which*

$$F^{\frac{1}{p}-\frac{1}{2}}_{p,r}(\mathbb{R};X) \hookrightarrow \gamma(L^2(\mathbb{R}),X).$$

From Proposition 3.4 one sees that if the embedding holds, then X has type p.

Problem 3.15. *Let $p \in [1,2)$. Characterize those Banach spaces X for which*

$$H^{\frac{1}{p}-\frac{1}{2}}_{p}(\mathbb{R}^d;X) \hookrightarrow \gamma(L^2(\mathbb{R}),X).$$

From Remark 3.11 one sees that if the embedding holds, then X has type p.

References

[1] F. Albiac and N.J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.

[2] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006, Reprint of the 1985 original.

[3] H. Amann, *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr. **186** (1997), 5–56.

[4] F. Bayart and S. Grivaux, *Invariant Gaussian measures for operators on Banach spaces and linear dynamics*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 1, 181–210.

[5] Z. Brzeźniak, J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis, *Itô's formula in UMD Banach spaces and regularity of solutions of the Zakai equation*, J. Differential Equations **245** (2008), no. 1, 30–58.

[6] Z. Brzeźniak, J. van Neerven, and D. Salopek, *Stochastic evolution equations driven by Liouville fractional Brownian motion*, Czechoslovak Math. J. **62(137)** (2012), no. 1, 1–27.

[7] D.L. Burkholder, *Martingales and singular integrals in Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 233–269.

[8] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.

[9] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[10] J. García-Cuerva, J. L. Torrea, and K. S. Kazarian, *On the Fourier type of Banach lattices*, Interaction between functional analysis, harmonic analysis, and probability (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, 1996, pp. 169–179.

[11] M. Girardi and L. Weis, *Operator-valued Fourier multiplier theorems on Besov spaces*, Math. Nachr. **251** (2003), 34–51.

[12] B.H. Haak and P.C. Kunstmann, *Admissibility of unbounded operators and wellposedness of linear systems in Banach spaces*, Integral Equations Operator Theory **55** (2006), no. 4, 497–533.

[13] M. Haase, *Transference principles for semigroups and a theorem of Peller.*, J. Funct. Anal. **261** (2011), no. 10, 2959–2998 (English).

[14] T. Hytönen, J. van Neerven, and P. Portal, *Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi*, J. Anal. Math. **106** (2008), 317–351.

[15] T.P. Hytönen, *Littlewood-Paley-Stein theory for semigroups in UMD spaces*, Rev. Mat. Iberoam. **23** (2007), no. 3, 973–1009.

[16] N.J. Kalton, J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis, *Embedding vector-valued Besov spaces into spaces of γ-radonifying operators*, Math. Nachr. **281** (2008), 238–252.
[17] N.J. Kalton and L. Weis, *The H^∞-functional calculus and square function estimates*, in preparation.
[18] M. Kemppainen, *On the Rademacher maximal function*, Studia Math. **203** (2011), no. 1, 1–31.
[19] C. Le Merdy, *γ-Bounded representations of amenable groups*, Adv. Math. **224** (2010), no. 4, 1641–1671.
[20] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes.
[21] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, Vol. II: Function Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer-Verlag, Berlin, 1979.
[22] J. Maas, *Malliavin calculus and decoupling inequalities in Banach spaces*, J. Math. Anal. Appl. **363** (2010), no. 2, 383–398.
[23] J. Maas and J.M.A.M. van Neerven, *A Clark-Ocone formula in UMD Banach spaces*, Electron. Commun. Probab. **13** (2008), 151–164.
[24] J. van Neerven, *Asymptotic behaviour of C_0-semigroups and γ-boundedness of the resolvent*, J. Math. Anal. Appl. **358** (2009), no. 2, 380–388.
[25] J. van Neerven, *γ-radonifying operators—a survey*, The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 44, Austral. Nat. Univ., Canberra, 2010, pp. 1–61.
[26] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis, *Conditions for stochastic integrability in UMD Banach spaces*, In: Banach Spaces and their Applications in Analysis (in Honor of Nigel Kalton's 60th Birthday), De Gruyter Proceedings in Mathematics, De Gruyter, 2007, pp. 127–146.
[27] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis, *Stochastic integration in UMD Banach spaces*, Ann. Probab. **35** (2007), no. 4, 1438–1478.
[28] J.M.A.M. van Neerven, M.C. Veraar, and L. Weis, *Stochastic maximal L^p-regularity*, Ann. Probab. **40** (2012), no. 2, 788–812.
[29] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis, *Maximal γ-regularity*, Preprint.
[30] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis, *Stochastic evolution equations in UMD Banach spaces*, J. Funct. Anal. **255** (2008), no. 4, 940–993.
[31] J.M.A.M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), no. 2, 131–170.
[32] J. Rosiński and Z. Suchanecki, *On the space of vector-valued functions integrable with respect to the white noise*, Colloq. Math. **43** (1980), no. 1, 183–201 (1981).
[33] H.-J. Schmeisser, *Vector-valued Sobolev and Besov spaces*, Seminar analysis of the Karl-Weierstraß-Institute of Mathematics 1985/86 (Berlin, 1985/86), Teubner-Texte Math., vol. 96, Teubner, Leipzig, 1987, pp. 4–44.
[34] H.-J. Schmeisser and W. Sickel, *Traces, Gagliardo-Nirenberg Inequalities and Sobolev Type Embeddings for Vector-valued Function Spaces*, Jenaer Schriften zur Mathematik und Informatik, 2001.
[35] H.-J. Schmeisser and W. Sickel, *Vector-valued Sobolev spaces and Gagliardo-Nirenberg inequalities*, Nonlinear elliptic and parabolic problems, Progr. Nonlinear Differential Equations Appl., vol. 64, Birkhäuser, Basel, 2005, pp. 463–472.
[36] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam, 1978.
[37] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
[38] H. Triebel, *Fractals and spectra*, Monographs in Mathematics, vol. 91, Birkhäuser Verlag, Basel, 1997, Related to Fourier analysis and function spaces.
[39] F. Zimmermann, *On vector-valued Fourier multiplier theorems*, Studia Math. **93** (1989), no. 3, 201–222.

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

E-mail address: M.C.Veraar@tudelft.nl